Republic of Iraq

Ministry of Higher Education and Scientific Research

Northern Technical University

Static Mechanics and Strength of Materials

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CONTENTS

Chapter 1

Introduction, Composition , Resolution of forces, and Resultants of Force Systems

1.1. Introduction

Mechanics: is a branch of the physical science that is concerned with the state of rest or motion of bodies that are subjected to the action of force. Objects of interest in sport biomechanics are human body and sport equipment. According to the nature of studied objects mechanics is divided into several branches (Figure 1).

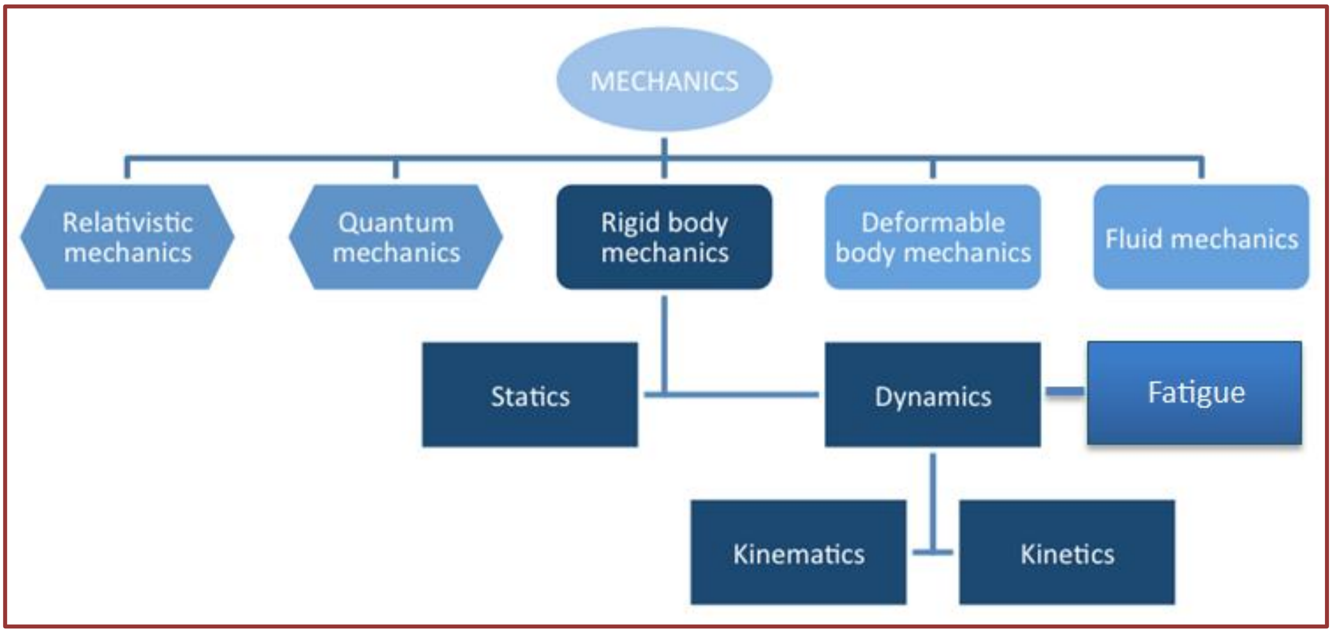


Figure 1. Branches of mechanics divided according to the nature of studied objects, and the division of rigid body mechanics

Static Mechanics

Statics is a branch of engineering mechanics that deals with the analysis of forces and interactions of bodies in equilibrium.

Dynamic Mechanics

Dynamic is concerned with the accelerated motion of bodies under effects of external forces.

Fatigue Mechanics

Fatigue is a failure mechanism that involves the cracking of materials and structural components due to cyclic (or fluctuating) stress.

1.2. Composition of Forces

The process of finding out the resultant force of a number of given forces is called the composition/compounding of forces.

1. Parallelogram Law

If two forces acting simultaneously on a particle is represented in magnitude and direction by two adjacent sides of a parallelogram, their resultant may be represented in magnitude and direction by the diagonal of the parallelogram which passes through the point of intersection, (figure 2).

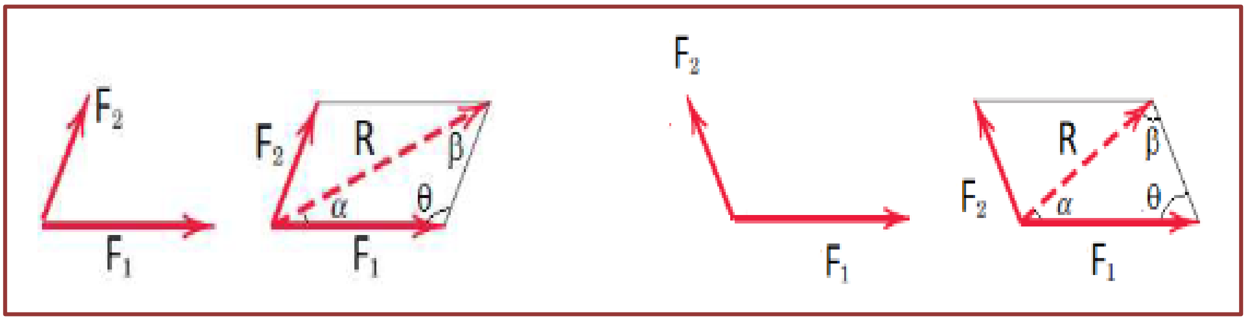


Figure 2. Parallelogram Law

The resultant of a pair of concurrent forces can be determined by the following equation:

$$Resultant, R = \sqrt{F_1^2 + F_2^2 - 2F_1 . F_2 cos\theta}$$

$$\alpha = \tan^{-1}\left(\frac{F_2^2\, sin\theta}{F_1^2 + F_2^2\, cos\theta}\right)$$

2. Triangle Law

Additionally, this equation can be used to determine the direction of the resultant or the unknown forces, (figure 3):

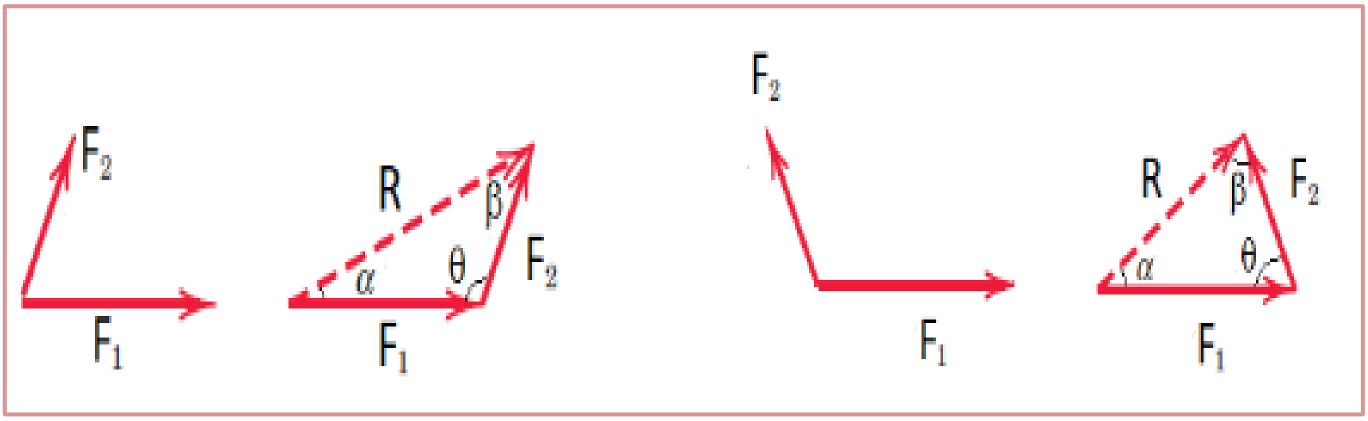


Figure 3. Triangle Law

$$\frac{R}{sin\theta} = \frac{F_1}{sin\beta} = \frac{F_2}{sin\alpha}$$

1.3. Resolution of a Force

The process of substituting a force by its components so that the net effect on the body remains the same is known as resolution of a force.

For each force, there exists an infinite number of possible sets of components.

Suppose a force is to be resolved into two components.

Then:

1. When one of the components is known, the second component can be obtained by applying the triangle rule.
2. When the line of action of each component is known, the magnitude and the sense of the components are obtained by parallelogram law.

1.4. Principle of Resolution

The algebraic sum of the resolved parts of a number of forces in the given direction is equal to the resolved part of their resultant in the same direction.

Replace a single force with its components through the process of resolution.

If a force (F) lies in the plane (x-y). The force (F) may be resolved into two rectangular components. The component of a force parallel to the x-axis is called the Horizontal component (F_x), and parallel to y-axis the is called Vertical component (F_y).

As an illustration of the following force analysis on two axes , (figure 4):

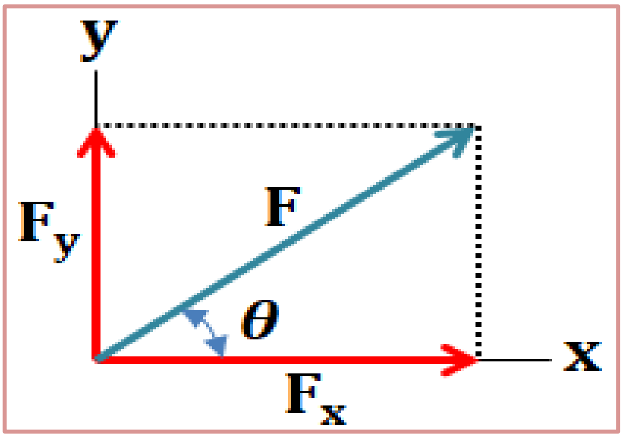


Figure 4. Force analysis on two axes

$$sin\theta = \frac{F_y}{F} \quad \Leftrightarrow \quad \boldsymbol{F_y = F\, sin\theta} \;\; \& \;\; cos\theta = \frac{F_x}{F} \quad \Leftrightarrow \quad \boldsymbol{F_x = F\, cos\theta}$$

$$R = \sqrt{F_x^2 + F_Y^2}$$

$$tan\theta = \frac{F_y}{F_x} \quad \Leftrightarrow \quad \boldsymbol{\theta = \tan^{-1}\left(\frac{F_y}{F_x}\right)}$$

1.5. Resultant of Force Systems

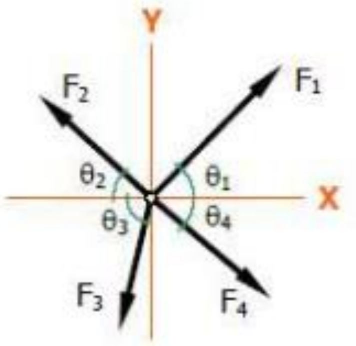
Resultant: Simplest force system which have same external effect of the original system.

1.5.1. Resultant of Coplanar Concurrent Force System

In x-y plane, the resultant of coplanar concurrent force system where the lines of action of all forces pass through a common point can be found by the following formulas:

$$R_x = \sum F_x \quad \rightarrow^+$$

$$R_x = F_{1x} - F_{2x} - F_{3x} + F_{4x}$$

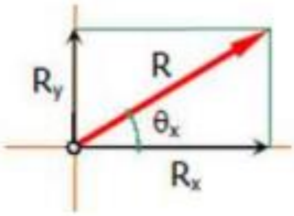


$$R_y = \sum F_y \quad \uparrow^+$$

$$R_y = F_{1y} + F_{2y} - F_{3y} - F_{4y}$$

$$R = \sqrt{{R_x}^2 + {R_y}^2}$$

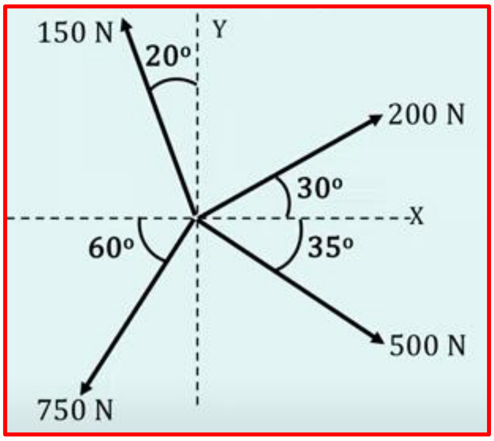
$$\theta_x = \tan^{-1}\left(\frac{R_y}{R_x}\right)$$

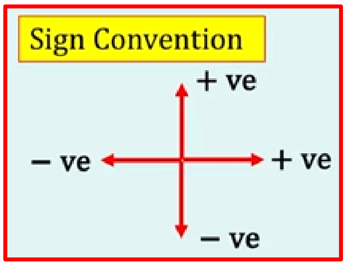


1.6. Solve examples

Example - 1

Calculate the magnitude and direction of resultant vector that is formed when taking the sum of the six forces shown below?





Solution:

First Method

$$\Sigma F_x = F_1.(\cos 30^\circ) - F_2.(\sin 20^\circ) - F_3.(\cos 60^\circ) + F_4.(\cos 35^\circ)$$

$$\Sigma F_x = 200 \times 0.866 - 150 \times 0.342 - 750 \times 0.5 + 500 \times 0.819$$
$$= 173.2 - 51.3 - 375 + 409.5 = \mathbf{156.4\ N}$$

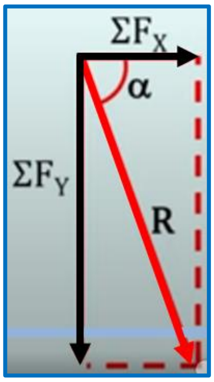
$$\Sigma F_y = F_1.(\sin 30^\circ) + F_2.(\cos 20^\circ) - F_3.(\sin 60^\circ) - F_4.(\sin 35^\circ)$$
$$\Sigma F_y = 200 \times 0.5 + 150 \times 0.94 - 750 \times 0.866 + 500 \times 0.574$$
$$= 100 + 141 - 649.5 - 287 = \mathbf{-695.5\ N}$$

Second Method

NO.	Description	ΣF_x (N)	ΣF_y (N)
1.	200 ∟30^o	$200 \cos 30^o = 173.21$	$200 \sin 30^o = 100$
2.	150 ∟110^o	$150 \cos 110^o = -51.3$	$150 \sin 110^o = 140.95$
3.	750 ∟240^o	$750 \cos 240^o = -375$	$750 \sin 240^o = -649.52$
4.	500 ∟325^o	$500 \cos 325^o = 409.58$	$500 \sin 325^o = -286.79$
Sum		**156.49**	**-695.34**

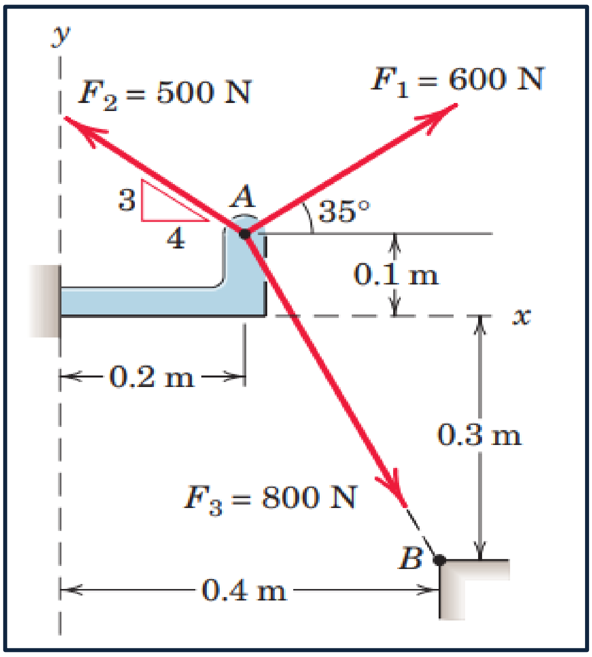
$$\mathbf{R} = \sqrt{\Sigma \mathbf{F}_x^2 + \Sigma \mathbf{F}_y^2} = \sqrt{156.4^2 + 695.5^2} = \mathbf{712.87\ N}$$

$$\boldsymbol{\theta} = \mathbf{tan}^{-1}\left(\frac{\mathbf{F_y}}{\mathbf{F_x}}\right) = \tan^{-1}\left(\frac{695.5}{156.4}\right) = \mathbf{77.33}^{\circ}$$



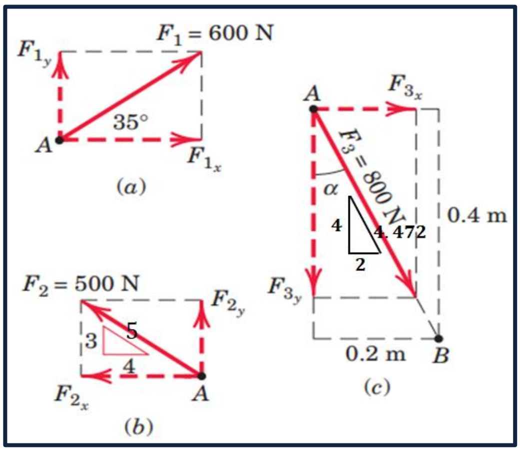
Example - 2

Calculate the magnitude and direction of resultant vector that is formed when taking the sum of the three forces act on point A, shown below?



Solution:

Draw free body diagram all forces, as the following.



First Method

From the figure:

$$\alpha = tan^{-1}\left(\frac{F_y}{F_x}\right) = tan^{-1}\left(\frac{0.2}{0.4}\right) = 26.565^{\circ}$$

$$\boldsymbol{\Sigma F_x} = F_1.(cos\,35^{\circ}) - F_2.(\frac{4}{5}) + F_3.\sin 26.565^{\circ}$$

$$\boldsymbol{\Sigma F_x} = 600 \times 0.819 - 500 \times 0.8 + 800 \times 0.447$$

$$= 491.4 - 400 + 375.6 = \boldsymbol{449\ N}$$

$$\boldsymbol{\Sigma F_y} = F_1.(sin\,35^{\circ}) + F_2.(\frac{3}{5}) - F_3.\cos 26.565^{\circ}$$

$$\boldsymbol{\Sigma F_y} = 600 \times 0.574 + 500 \times 0.6 - 800 \times 0.894$$

$$= 344.4 + 300 - 715.2 = \boldsymbol{-70.8\ N}$$

Second Method

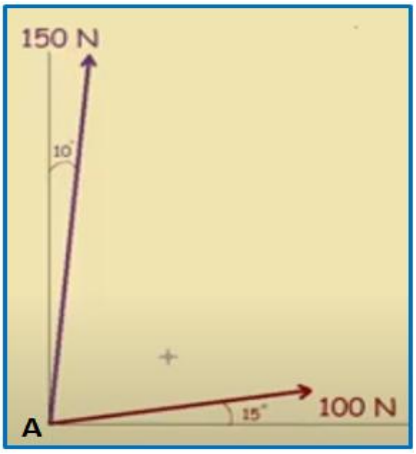
NO.	Description	ΣF_x (N)	ΣF_y (N)
1.	$600\ \llcorner 35^o$	$600\cos 35^o = 491.49$	$600\sin 35^o = 344.4$
2.	$500\ \llcorner 143.13^o$	$500\cos 143.13^o = -400$	$500\sin 143.13^o = 300$
3.	$800\ \llcorner 296.565^o$	$800\cos 296.565^o = 357.77$	$800\sin 296.565^o = -715.2$
Sum		**449.26**	**-70.8**

$$\boldsymbol{R} = \sqrt{\boldsymbol{\Sigma F_x^2 + \Sigma F_y^2}} = \sqrt{449^2 + (-74.8)^2} = \boldsymbol{455.188\ N}$$

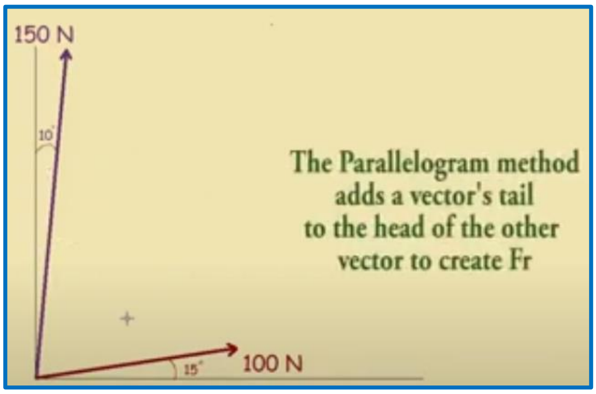
$$\boldsymbol{\theta} = \mathbf{tan^{-1}}\left(\frac{\boldsymbol{F_y}}{\boldsymbol{F_x}}\right) = \tan^{-1}\left(\frac{-70.8}{449}\right) = \boldsymbol{8.96^{\circ}}$$

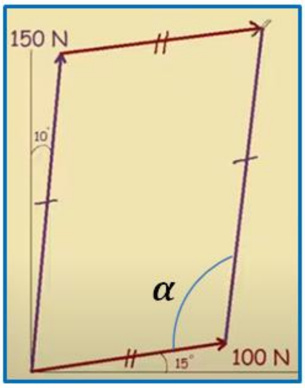
Example - 3

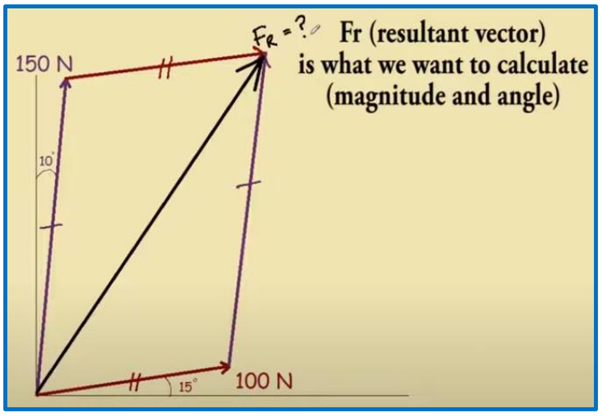
Calculate the magnitude and direction of resultant vector that is formed when taking the sum of the two forces act on point A, shown below?

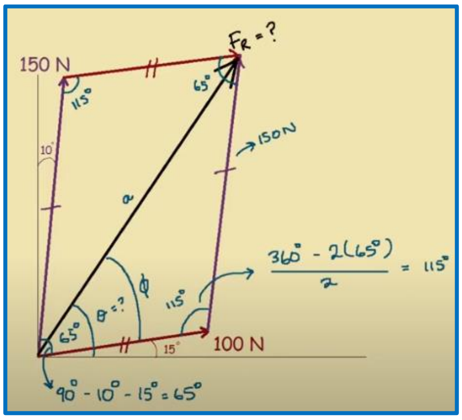


Solution:









$$Resultant, R = \sqrt{F_1^2 + F_2^2 - 2F_1F_2cos\alpha}$$

$$R = \sqrt{100^2 + 150^2 - 2 \times 100 \times 150 \times \cos 115^{\circ}} = 212.55\ N$$

From Sine Rule

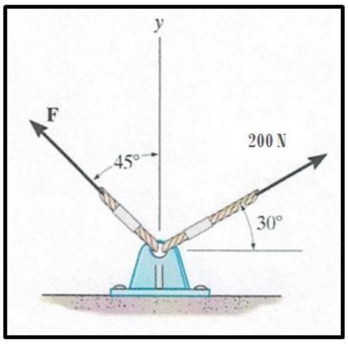
$$\frac{R}{\sin\theta} = \frac{F_1}{\sin\beta} = \frac{F_2}{\sin\alpha}$$

$$\frac{212.55}{\sin 115} = \frac{150}{\sin\emptyset}$$

$$\emptyset = \sin^{-1}\frac{150\sin 115}{212.55} = \sin^{-1}(0.64) = 39.79^{o}$$

Example - 4

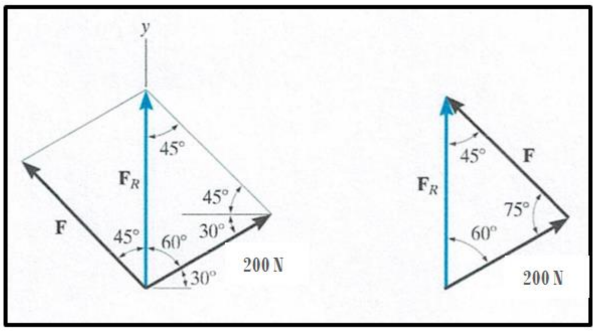
Calculate the magnitude of a force (F). also magnitude and direction of resultant vector that is formed when taking the sum of the two shown below?



Solution:

From Sine Rule

$$\frac{R}{sin\theta} = \frac{F_1}{sin\beta} = \frac{F_2}{sin\alpha}$$



$$\frac{F}{\sin 60^\circ} = \frac{200}{\sin 45^\circ}$$

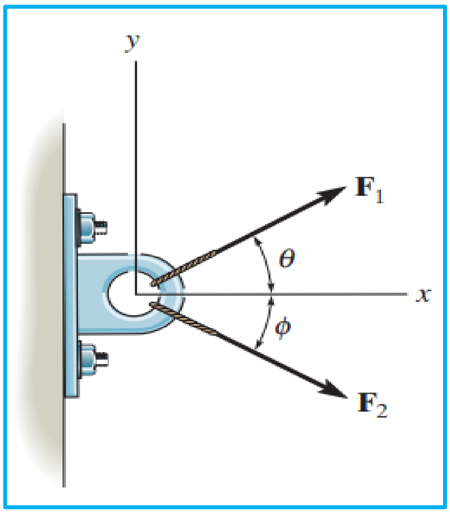
$$F = 245 \text{ N}$$

$$\frac{F_R}{\sin 75^\circ} = \frac{200}{\sin 45^\circ}$$

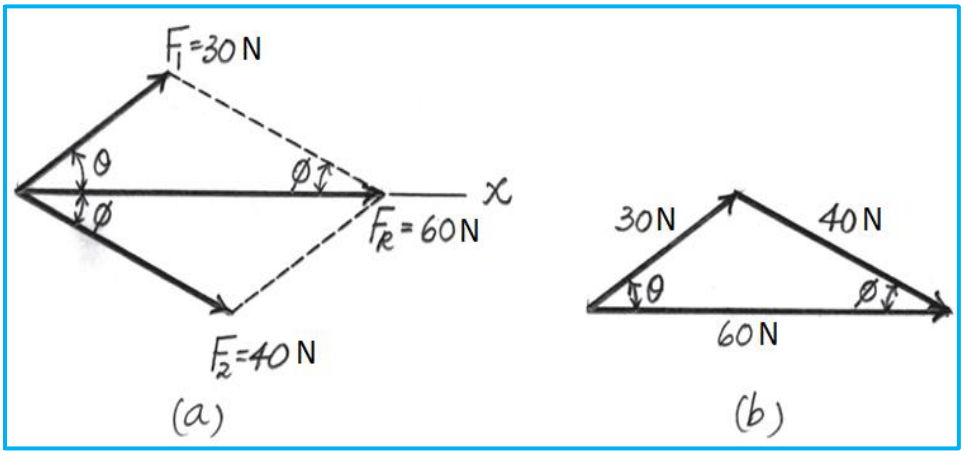
$$F_R = 273 \text{ N}$$

Example - 5

If F_1 = 30 N and F2 = 40 N, determine the angles u and f so that the resultant force is directed along the positive x axis and has a magnitude of F_R = 60 N.



Solution:



Parallelogram Law. The parallelogram law of addition is shown in Fig. *a*.
Trigonometry. Applying the law of cosine by referring to Fig. *b*,

$$40^2 = 30^2 + 60^2 - 2(30)(60)\cos\theta$$

$$\theta = 36.34° = 36.3°$$

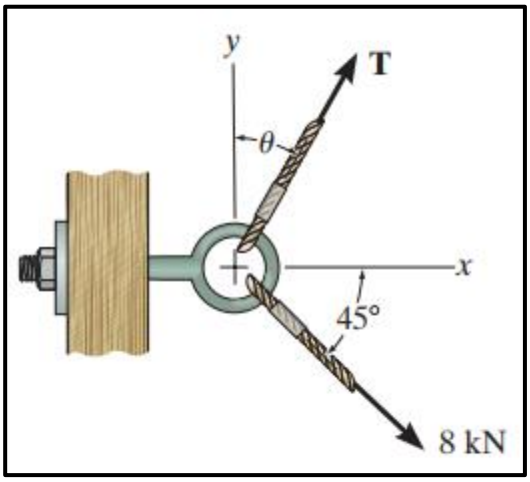
And

$$30^2 = 40^2 + 60^2 - 2(40)(60)\cos\phi$$

$$\phi = 26.38° = 26.4°$$

1.7. Chapter Questions

1. If the magnitude of the resultant force is to be (9 KN) directed along the positive x - axis, determine the magnitude of force (T) acting on the eyebolt and its angle.



{*Results:* $T = 6.57\ KN\ ;\ \theta = 30.6^{\circ}\ ;\ \emptyset = 75.6^{\circ}$}

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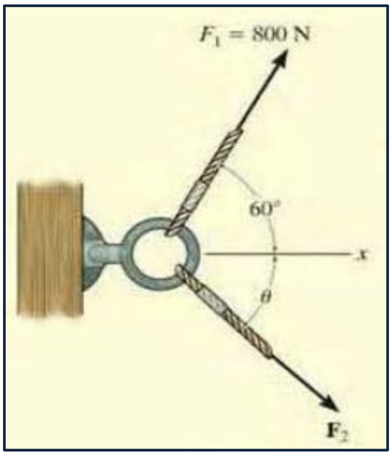
2. It is required that the resultant force acting on the eyebolt in Figure be directed along the positive axis and that (F2) have a minimum magnitude. Determine this magnitude. the angle (θ), and the corresponding resultant force.



{*Results:* $T = 6.57\ KN\ ;\ \theta = 90^{\circ}$}

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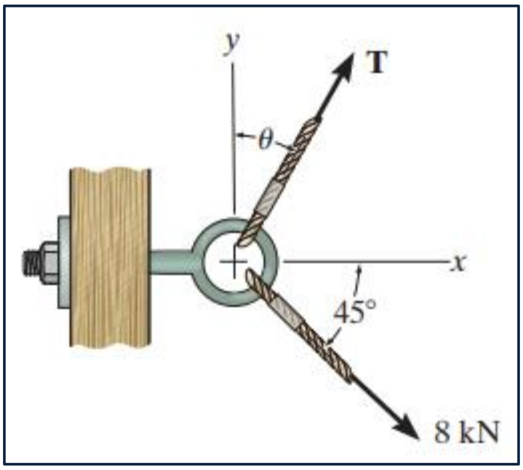
3. If ($\theta = 30^{\circ}$) and ($\mathrm{T} = 6\ \mathrm{KN}$) , determine the magnitude of the resultant force acting on the eyebolt and its direction measured clockwise from the positive x axis.



{*Results:* $F_R = 8.67\ KN\ ; \alpha = 63.05^{\circ}\ ;\ \emptyset = 3.05^{\circ}$}

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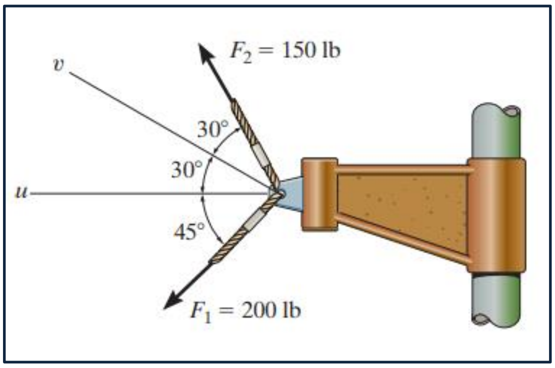
4. Determine the magnitude of the resultant force acting on the bracket and its direction measured counterclockwise from the positive u axis.



{*Results:* $F_R = 217\ N\ ; \alpha = 63.05^{\circ}\ ;\ \emptyset = 3.05^{\circ}$}

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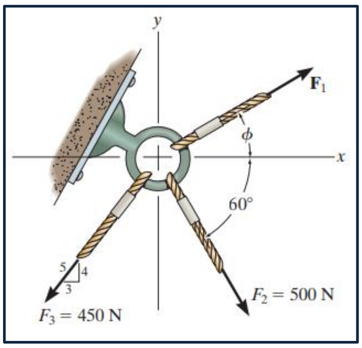
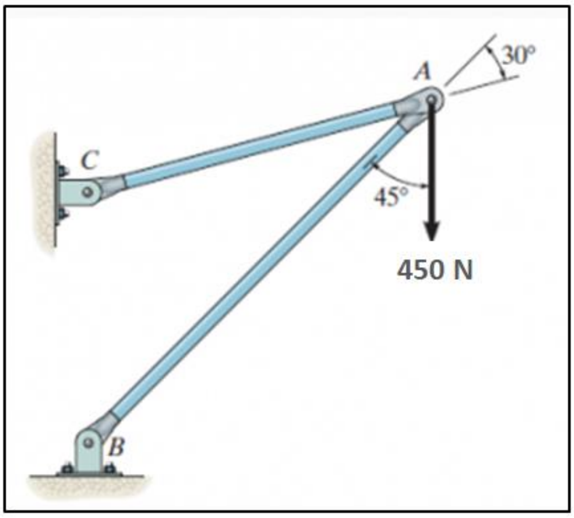
5. If ($F_1 = 600\ N$) and ($\emptyset = 30^{\circ}$), determine the magnitude of the resultant force acting on the eyebolt and its direction measured clockwise from the positive x axis



{*Results:* $F_R = 701.91\ N\ ; \theta = 44.06^{\circ}$}

6. The force (F = 450 N) acts on the frame. Resolve this force into components acting along members AB and AC , and determine the magnitude of each component.



{*Results:* $F_{AB} = 86\ N$; $F_{AC} = 636\ N$ }

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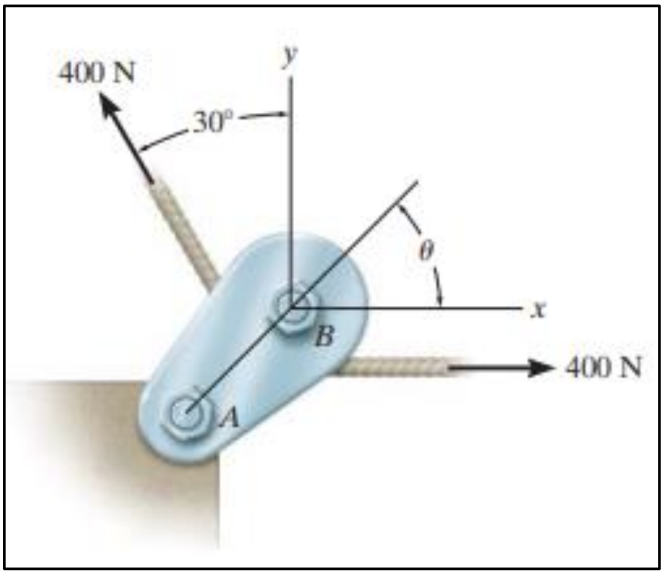
7. If the tension in the cable is 400 N, determine the magnitude and direction of the resultant force acting on the pulley. This angle is the same angle of line AB on the tailboard block.



{*Results:* $R = 400\ N$; $\theta = 60^o$}

Chapter 2

Moment of a Forces

and

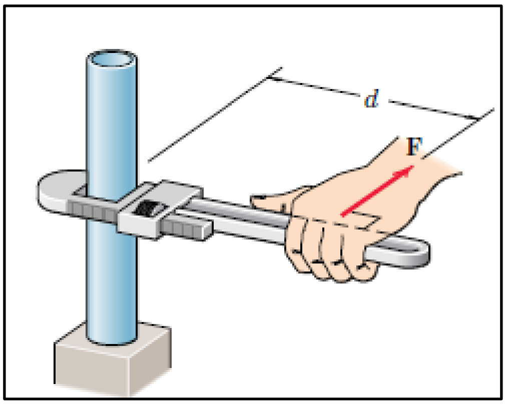
Moment of a couples

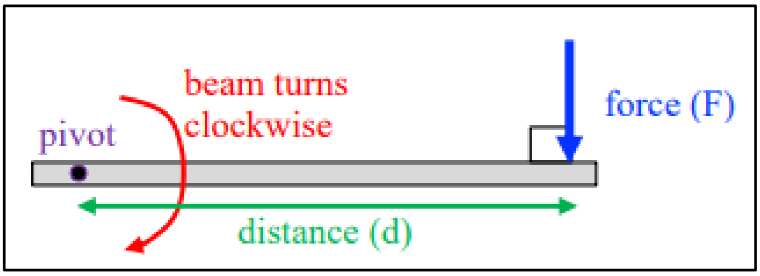
2.1. Introduction

In the previous chapter, we have been discussing the effects of forces, acting on a body, through their lines of action or at the point of their intersection. But in this chapter, we shall discuss the effects of these forces, at some other point, away from the point of intersection or their lines of action

2.2. Moment of a Force

Moment is ability of the force to produce twisting or turning a body about an axis.





$$M = F \cdot d$$

Where:

$\boldsymbol{M}$: $The\ moment\ of\ the\ force\ (N.m).$

$\boldsymbol{F}$: $Applied\ force\ (N).$

$\boldsymbol{d}$: $is\ the\ perpendicular\ distance\ from\ the\ axis\ moment$

$\quad to\ the\ line\ of\ action\ of\ the\ force\ (m).$

$$Units: kN.m, N.m, N.mm$$

Sign Convention:

<u>Note:</u> Always taking clockwise as positive moment.

2.3. Principle of moments

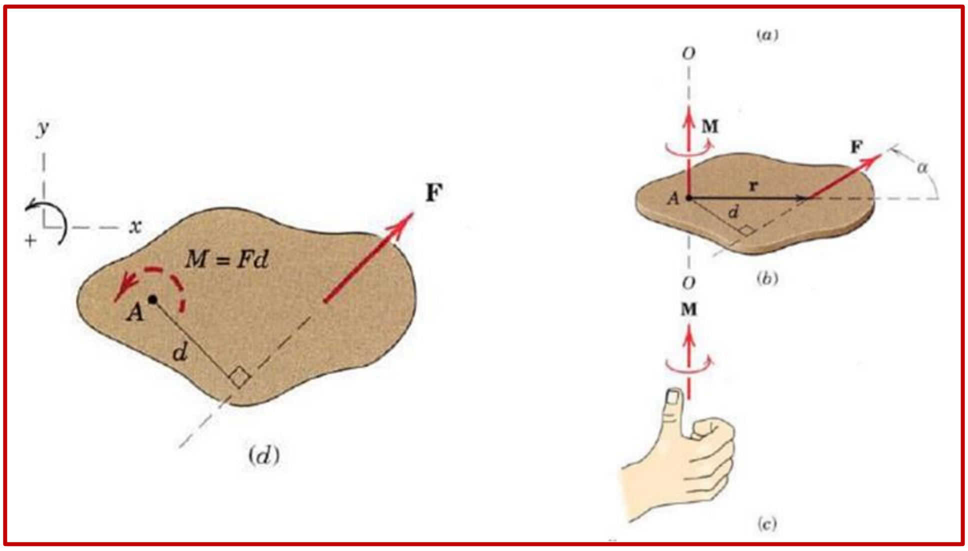
The moment of a force with respect to any axis (or point) is equal to the algebraic sum of the moments of its components with respect to the same axis.

$$M = \sum F \cdot d$$

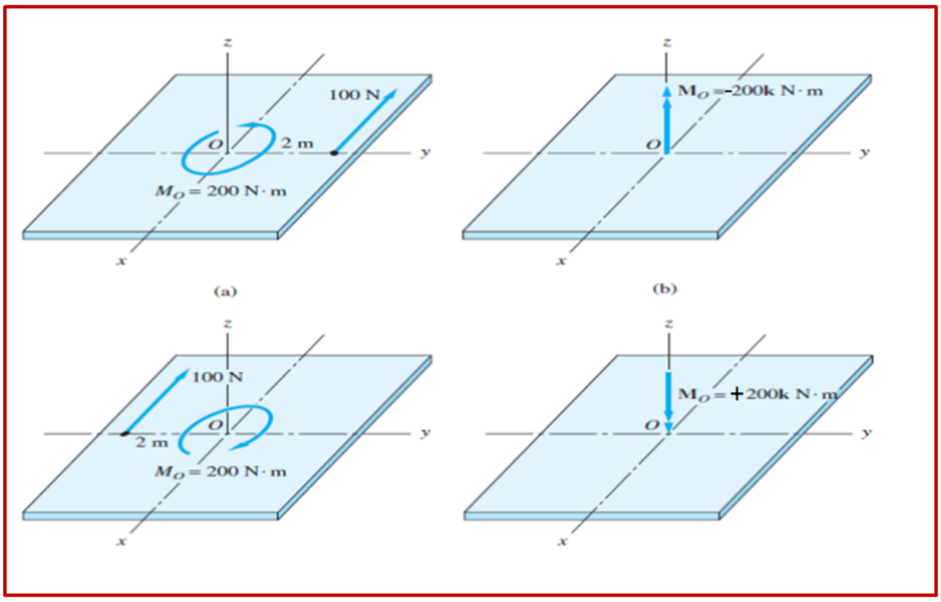
Where:

Is the moment arm, which is the perpendicular distance from the axis of rotation to the line of action of the force.

$$M = F.r.sin \propto$$



2.3.1. Moment's Direction

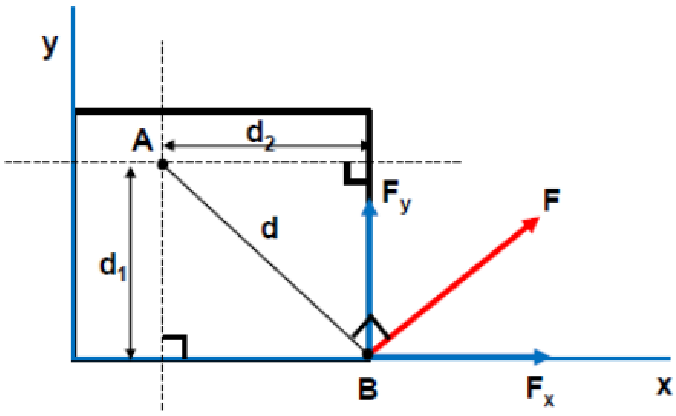


2.3.2. Varignon's Theorem

One of the most useful principles of mechanics is Varignon's theorem, which states that the moment of a force about any point is equal to the sum of the moments of the components of the force about the same point .

$$M_A = F.d$$

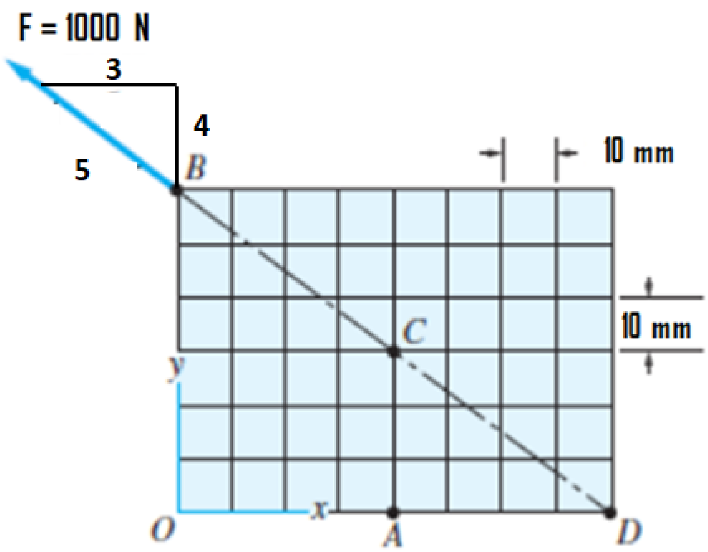
$$M_A = F_x.d_1 + F_Y.d_2$$



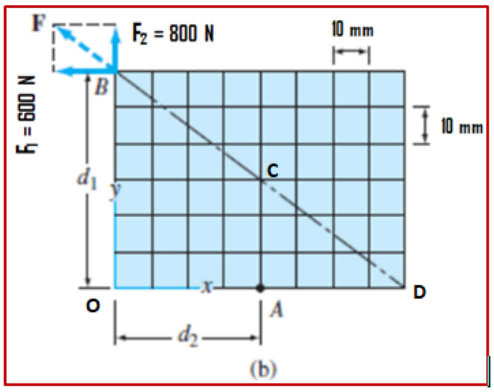
2.4. Solve examples

Example - 1

Determine the moment of the force (F) in Figure below about points (A), (C) , (D) and (O) ?



Solution



$$\circlearrowright +M_A = F_1.d_1 - F_2.d_2$$
$$\circlearrowright +M_A = -600 \times 60 + 800 \times 40 = -4000\ N.mm = 4000\ N.mm \circlearrowleft$$

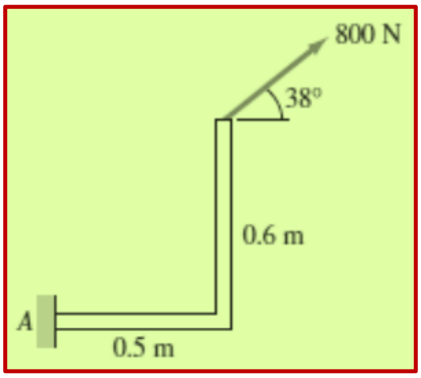
$$\circlearrowright +M_C = F_1.d_1 - F_2.d_2$$
$$\circlearrowright +M_C = -600 \times 30 + 800 \times 4000 = 18000\ N.mm \circlearrowright$$

$$\circlearrowright +M_D = F_1.d_1 - F_2.d_2$$
$$\circlearrowright +M_D = -600 \times 60 + 800 \times 80 = 28000\ N.mm \circlearrowright$$

$$\circlearrowright +M_O = F_1.d_1 - F_2.d_2$$
$$\circlearrowright +M_O = -600 \times 60 + 0 = -36000\ N.mm = 36000\ N.mm \circlearrowleft$$

Example - 2

Determine the magnitude and sense of the moment of the (F = 800 N) force about point (A)?

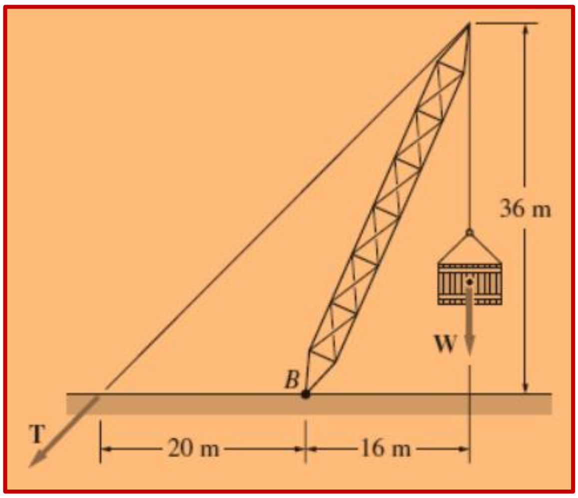


Solution

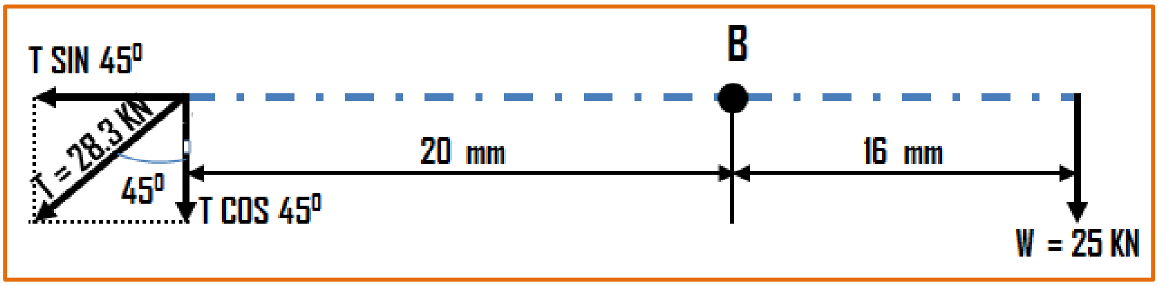
$$\circlearrowright +M_A = F_x \times 0.6 - F_y \times 0.5$$
$$= (800 \cos 38^o) \times 0.5 - (800 \sin 38^o) \times 0.6$$
$$M_A = 378.25 - 246.26 = 131.99\ N.m$$
$$\therefore\ M_A = 131.99\ N.m\ \circlearrowright$$

Example – 3

Given that (T =28.3 KN) and (W =25 KN), determine the magnitude and sense of the moments about point (B) of the following: (a) the force T; (b) the force W; and (c) forces T and W combined?



Solution



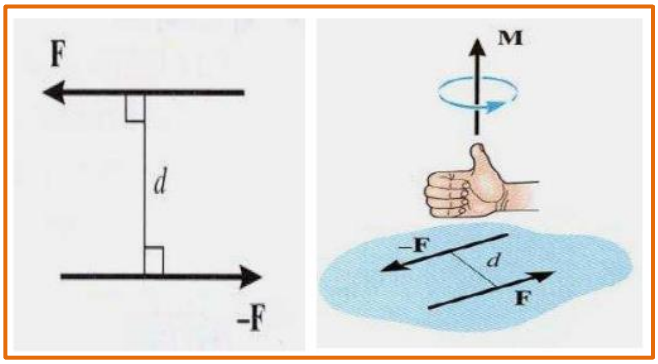
(a). For T: $\circlearrowright +M_B = -20\ (28.3\ \cos 45^o) = -400\ = 400\ KN.mm \circlearrowleft$

(b). For W: $\circlearrowright +M_B = -25\ (16) = -400 = 400\ KN.mm \circlearrowleft$

(c). For T & W: $\circlearrowright \sum M_B = 400 - 400 = 0$

2.5. Moment of a Couple

A couple is defined as two parallel forces with the same magnitude but opposite in direction separated by a perpendicular distance (d).



The moment of a couple is defined as:

$$M_O = F \, . d$$

(Using a scalar analysis) or as.

$$M_O = r \, . F$$

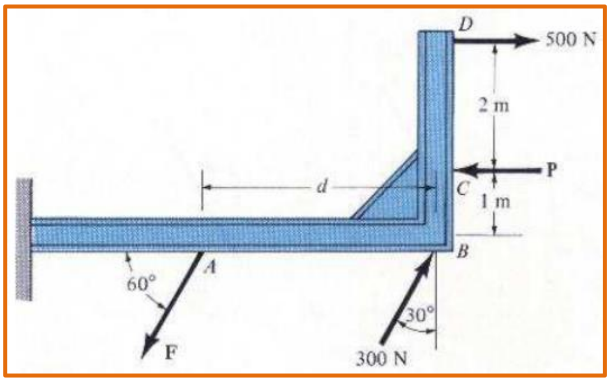
(Using a vector analysis). Here r is any position vector from the line of action of −F to the line of action of F.

The net external effect of a couple is that the net force equals zero and the magnitude of the net moment equals (F . d) Since the moment of a couple depends only on the distance between the forces, the moment of a couple is a free vector. It can be moved anywhere on the body and have the same external effect on the body. Moments due to couples can be added using the same rules as adding any vectors.

Two couples act on the beam. One couple is formed by the forces at A and B, and other by the forces at C and D. If the resultant couple is zero, determine the magnitudes of P and F, and the distance d between A and B.

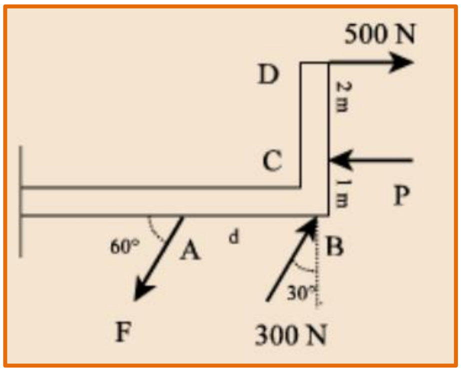
Example – 4

Two couples act on the beam. One couple is formed by the forces at A and B, and other by the forces at C and D. If the resultant couple is zero, determine the magnitudes of P and F, and the distance d between A and B.



Solution

Free Body Diagram (F. B. D) of the figure.



Since these are couples we must have:

$F = 300$ N

$P = 500$ N

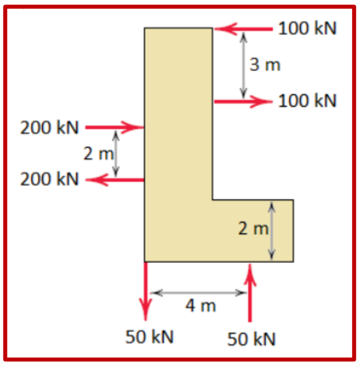
The resultant coupe is:

$M = +500 * 2 - 300 * d \cos 30^0 = 0$

Thus d = 3.85 m

Examples - 5

Determine the resultant moment of the three couples acting on the plate?



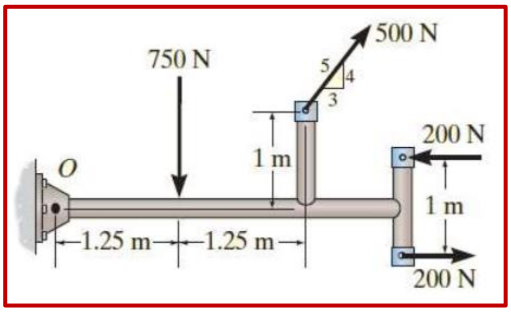
Solution

$$\circlearrowright +M = \Sigma F.d = 200 * 2 - 100 * 3 - 50 * 4 = -100\ KN.m$$

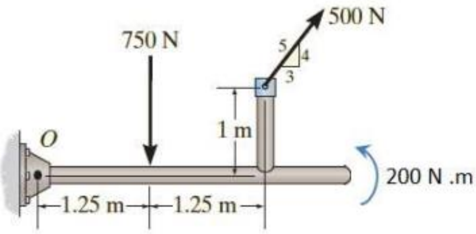
$$= 100\ KN.m\ \circlearrowleft\ Anticlockwise$$

Example - 6

Determine the resultant moment with respect to point (O)?



Solution



$$M_{couple} = F \cdot d = 200 \times 1 = 200\ N.m$$

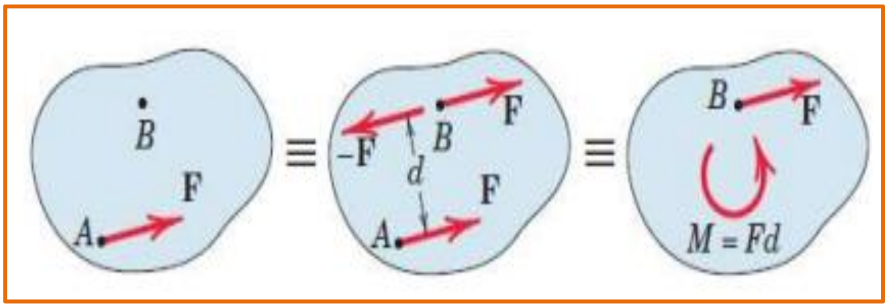
$$M_o = \sum F \cdot d$$

$$M_o = 750 \times 1.25 + 500 \times \frac{3}{5} \times 1 - 500 \times \frac{4}{5} \times 2.5 - 200$$

$$M_o = 37.5\ N.m$$

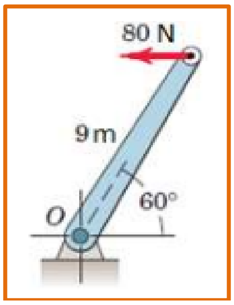
2.6. Force – Couple Systems

According to the principle of transmissibility, the force can be moved to any point along its line of action, as it produces the same effect on the body. However, if we want to move the force to a point not lying on its line of action, it must generate a couple such that it produces the same effect as the force. This is known as forcecouple system. The replacement of a force into a force and a couple is explain in Figure below, where the given force F acting at point A is replaced by an equal force F at point B and the counterclockwise couple $M = Fd$.



Example – 7

Replace the horizontal (80 N) force acting on the lever by an equivalent system consisting of a force at (O) and a couple.

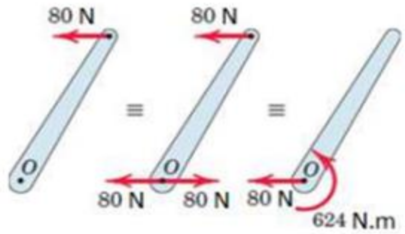


Solution

$$M_o = F \cdot d$$

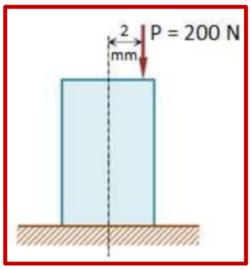
$$M_o = 80 \times (9 \ \sin 60)$$

$$M_o = 624 \ N.m$$



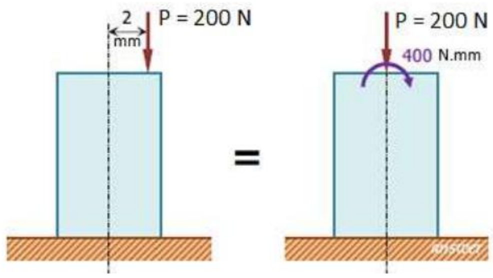
Example - 8

For the compression member shown in the figure, replace the force (P = 200 N) by an equivalent axial load and a couple.



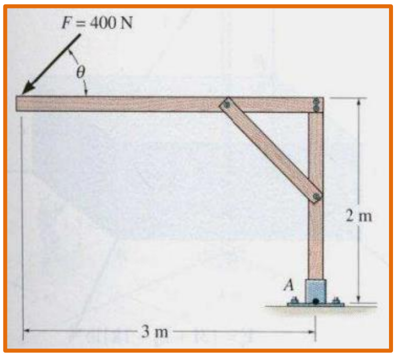
Solution

$$M_{couple} = F \cdot d = 200 \times 2 = 400 \ N.mm$$



2.7. Chapter Questions

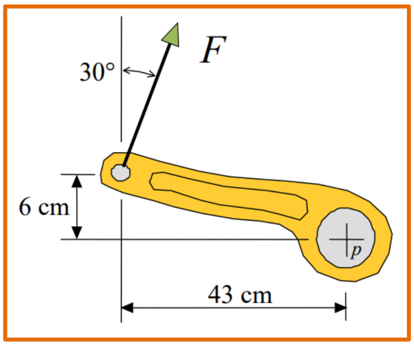
1. A 400 N force is applied to the frame and ϴ = 20°, as in the following figure. Find the moment of the force at A?



{Answer: $M_B = 1160\ N.m$}

---

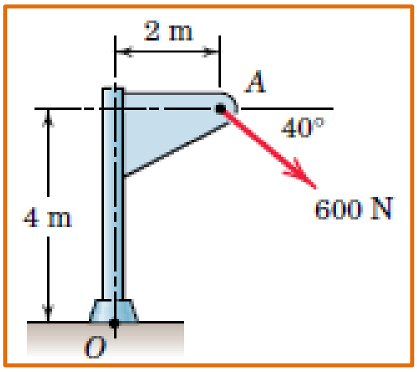
1. The wrench shown is used to turn drilling pipe. If a torque (moment) of (800 N.m) about point (p) is needed to turn the pipe, determine the required force (F).



{Answer: $F = 239\ N$}

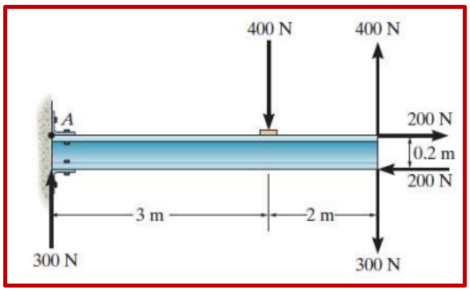
---

2. Calculate the moment about the base point (O) of the (600 N)?



{Answer: $M_o = 2610\ N.m$}

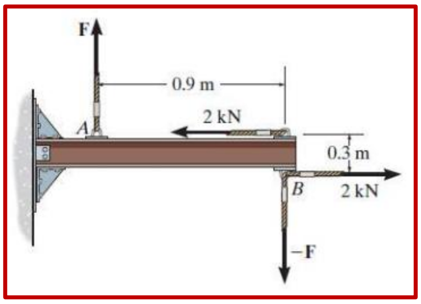
3. Determine the resultant moment acting on the beam?



{Answer: $M_{Couple} = 740\ N.m$ ↻}

---

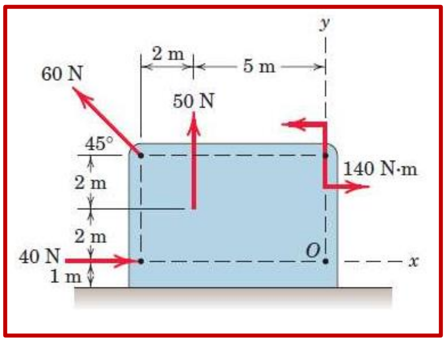
4. Determine the magnitude of (F), so that the resultant moment acting on the beam is (1.5 kN.m) clockwise.?



{Answer: $F = 2.333\ KN$}

---

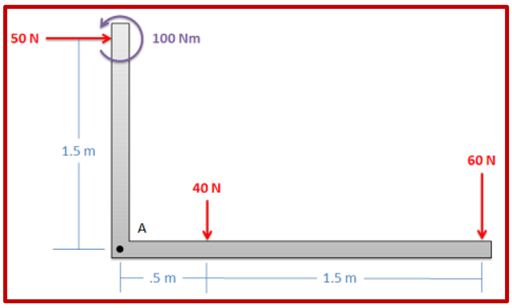
5. Determine the resultant moment of the three forces and one couple which act on the plate shown about point (O)?



{Answer:: $M_O = 237279\ N.m$ ↻}

---

6. Find the equivalent force couple system about point A for the set of forces shown below?



{Answer:: $M_A = 115\ N.m$ ↺}

Chapter 3

Equilibrium

3.1. Equilibrium

When all the sums of the forces of the system in certain directions and the sum of moments of the forces with respect to certain axes are zero for any particular force system, its resultant is zero, and the body on which the system acts is in equilibrium. The conditions assuring equilibrium of a body with a particular type of force system can therefore be expressed as a set of algebraic equations which must be satisfied. By means of these conditions, it is possible to determine one or more unknown forces or reactions acting on a body which is in equilibrium.

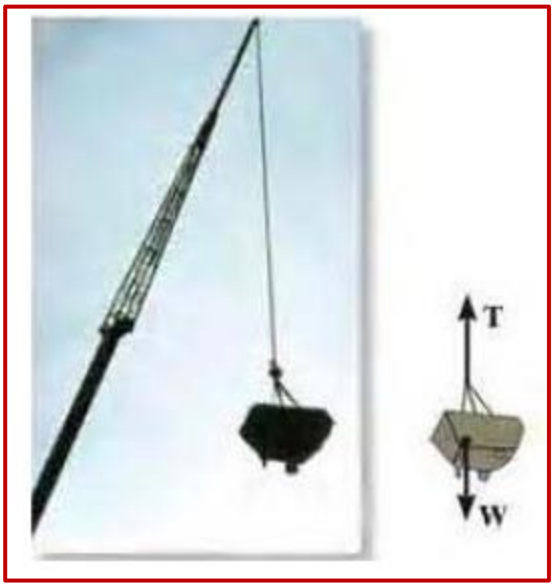
3.2. Free body Diagrams

Is a sketch of a body, a portion of a body, or two or more bodies completely isolated or free from all other bodies, showing the forces exerted by all other bodies on the one being considered.

Procedure for drawing a free body diagram

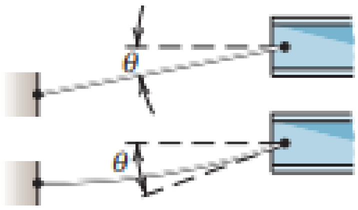
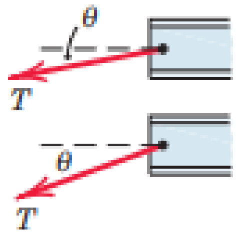
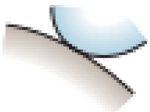
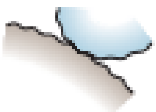
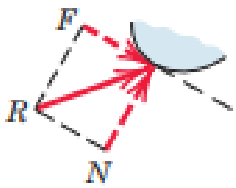
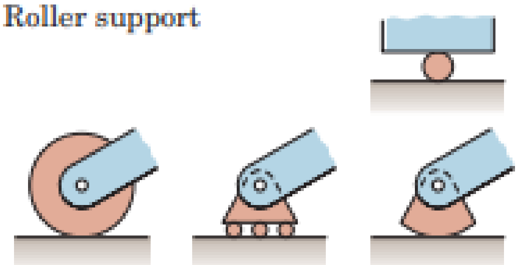
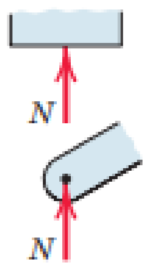
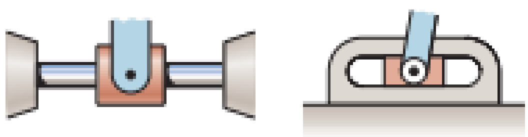
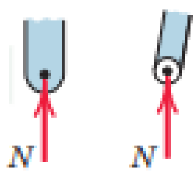
1. **Draw outlined shape**
   Imagine the particle to be isolated or cut (free) from its surroundings by drawing its outlined shape.
2. **Show all forces**
   Indicate on this sketch all the forces that acts on the particle. These forces can be active forces, which tend to set the particle in motion. Or they can be reactive forces which are the result of the constraints or support that tend to prevent motion.

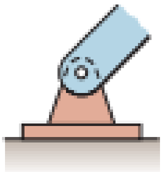
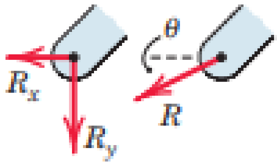
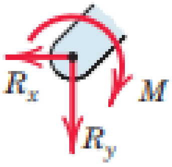
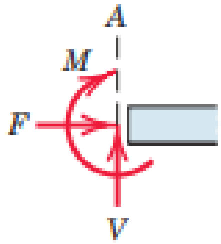
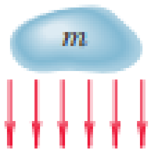
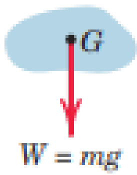
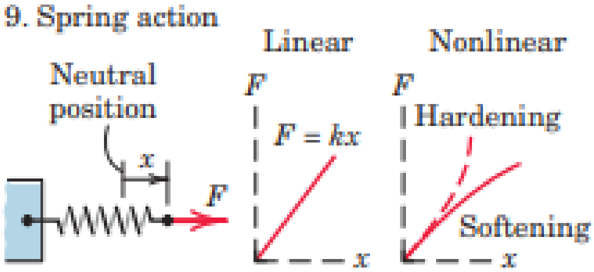
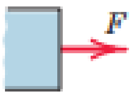
The forces that are known should be labeled with their proper magnitudes and directions. Letters are used to represent the magnitudes and directions of forces that are unknown.



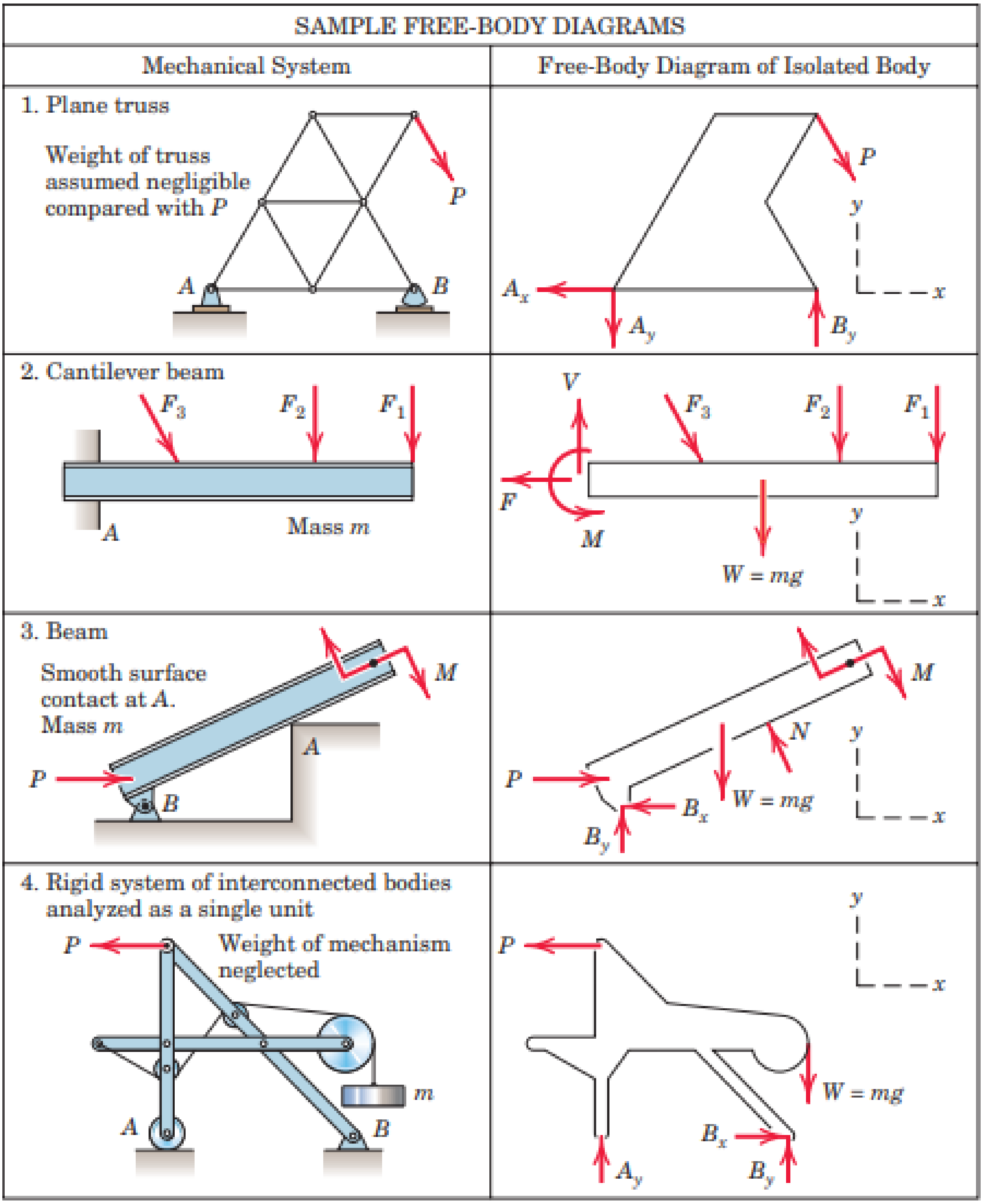
Note

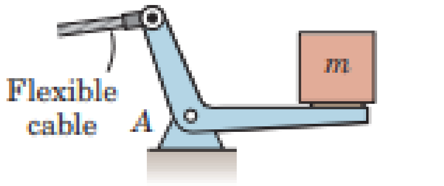
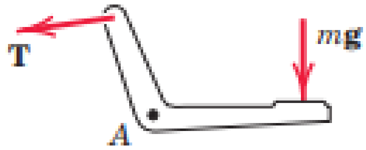
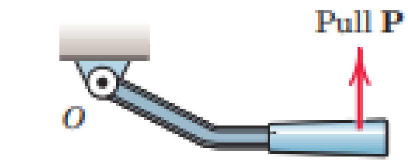
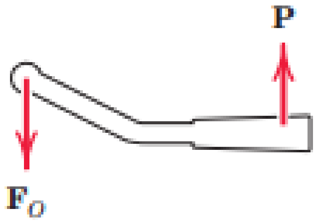
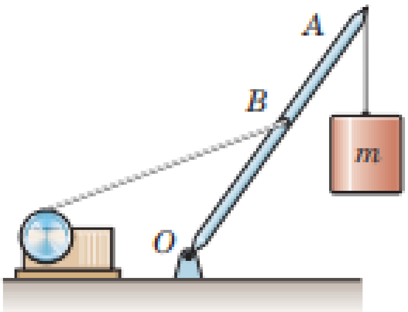
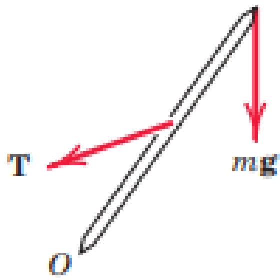
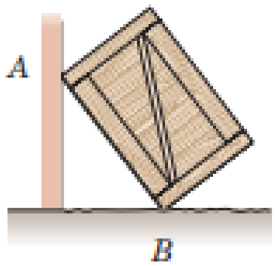
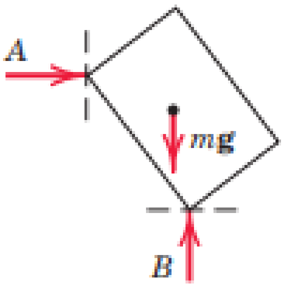
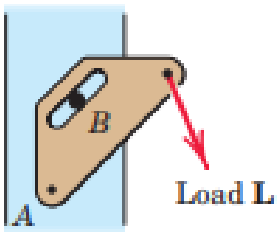
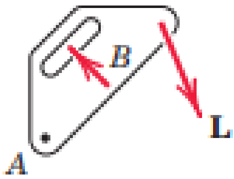
All cables will be assumed to have negligible weight and they cannot stretch. Also, a cable can support only a tension or pulling force and this force always acts in the direction of the cable.

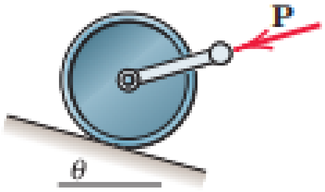
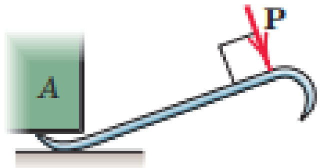
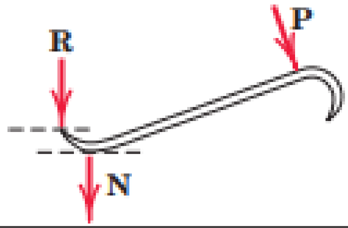
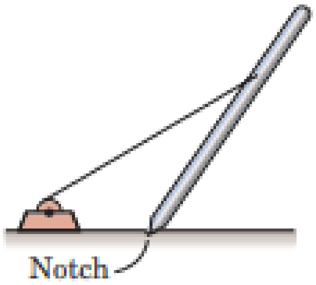
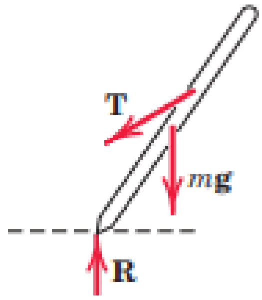
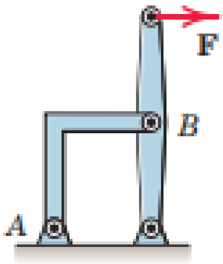
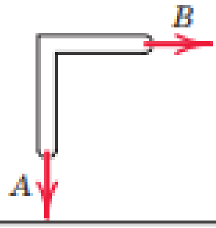
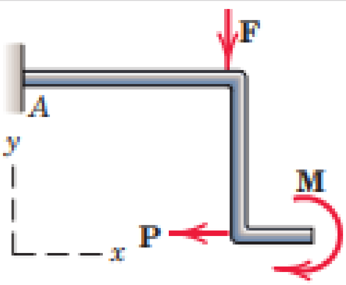
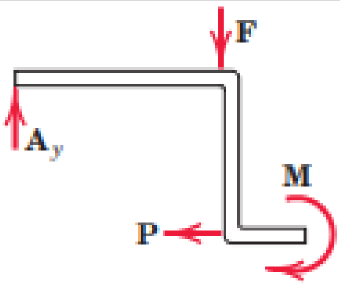
MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS	
Type of Contact and Force Origin	**Action on Body to Be Isolated**
1. Flexible cable, belt, chain, or rope Weight of cable negligible Weight of cable not negligible 	 Force exerted by a flexible cable is always a tension away from the body in the direction of the cable.
2. Smooth surfaces 	 Contact force is compressive and is normal to the surface.
3. Rough surfaces 	 Rough surfaces are capable of supporting a tangential component F (frictional force) as well as a normal component N of the resultant contact force R.
4. Roller support 	 Roller, rocker, or ball support transmits a compressive force normal to the supporting surface.
5. Freely sliding guide 	 Collar or slider free to move along smooth guides; can support force normal to guide only.

MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS (*cont.*)	
Type of Contact and Force Origin	**Action on Body to Be Isolated**
6. Pin connection 	Pin free to turn  Pin not free to turn  A freely hinged pin connection is capable of supporting a force in any direction in the plane normal to the pin axis. We may either show two components R_x and R_y or a magnitude R and direction θ. A pin not free to turn also supports a couple M.
7. Built-in or fixed support 	 A built-in or fixed support is capable of supporting an axial force F, a transverse force V (shear force), and a couple M (bending moment) to prevent rotation.
8. Gravitational attraction 	 The resultant of gravitational attraction on all elements of a body of mass m is the weight $W = mg$ and acts toward the center of the earth through the center mass G.
9. Spring action 	 Spring force is tensile if spring is stretched and compressive if compressed. For a linearly elastic spring the stiffness k is the force required to deform the spring a unit distance.

Examples of Free Body Diagrams

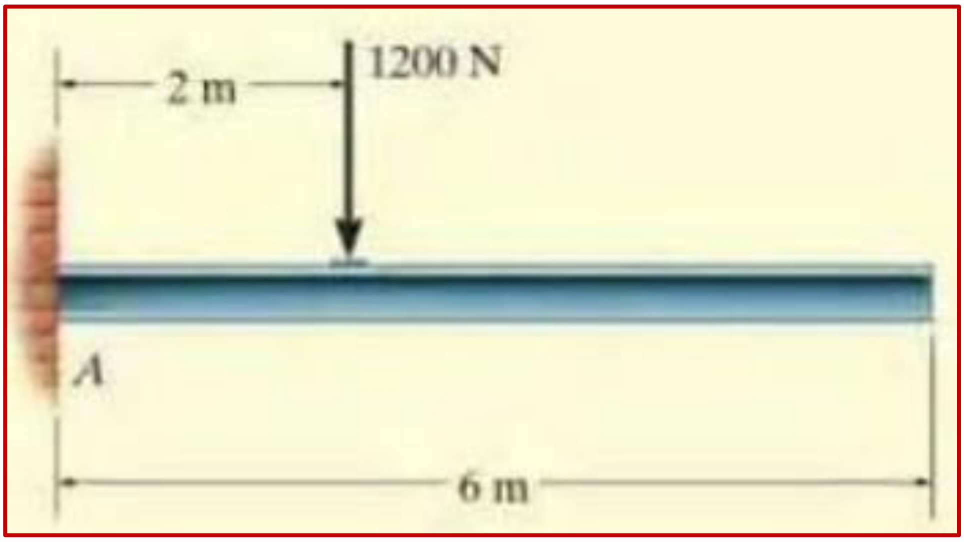


<table>
<tr><th></th><th>Body</th><th>Incomplete FBD</th></tr>
<tr><td>1. Bell crank supporting mass m with pin support at A.</td><td></td><td></td></tr>
<tr><td>2. Control lever applying torque to shaft at O.</td><td></td><td></td></tr>
<tr><td>3. Boom OA, of negligible mass compared with mass m. Boom hinged at O and supported by hoisting cable at B.</td><td></td><td></td></tr>
<tr><td>4. Uniform crate of mass m leaning against smooth vertical wall and supported on a rough horizontal surface.</td><td></td><td></td></tr>
<tr><td>5. Loaded bracket supported by pin connection at A and fixed pin in smooth slot at B.</td><td></td><td></td></tr>
</table>

	Body	Wrong or Incomplete FBD
1. Lawn roller of mass m being pushed up incline θ.		
2. Prybar lifting body A having smooth horizontal surface. Bar rests on horizontal rough surface.		
3. Uniform pole of mass m being hoisted into position by winch. Horizontal supporting surface notched to prevent slipping of pole.		
4. Supporting angle bracket for frame; pin joints.		
5. Bent rod welded to support at A and subjected to two forces and couple.		

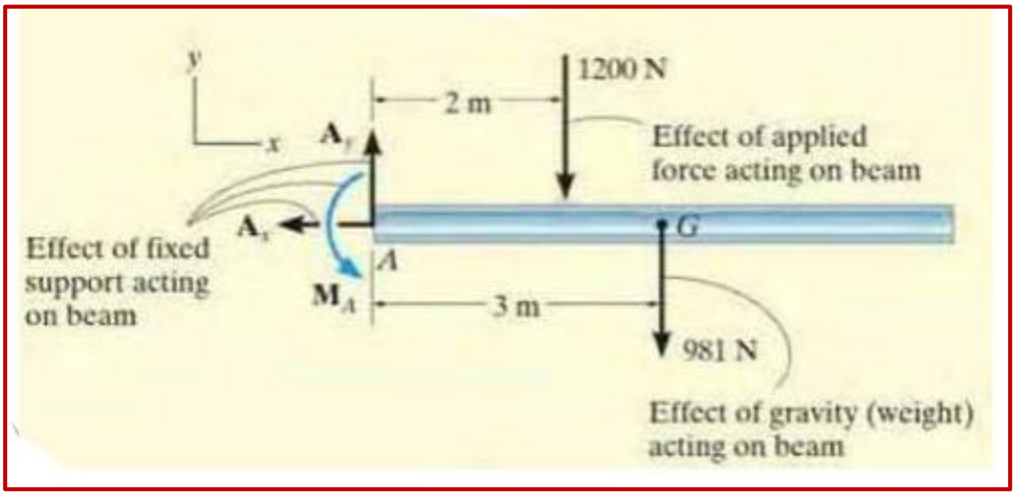
Example - 1

Draw the free body diagram of the uniform beam shown in figure. The beam has a mass of (100 kg)?



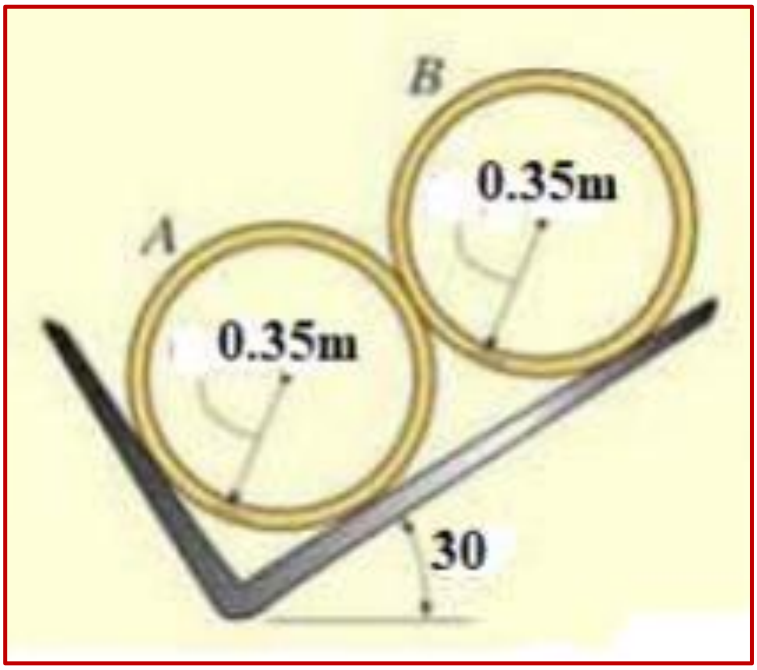
Solution:

Free Body Diagram (F. B. D)



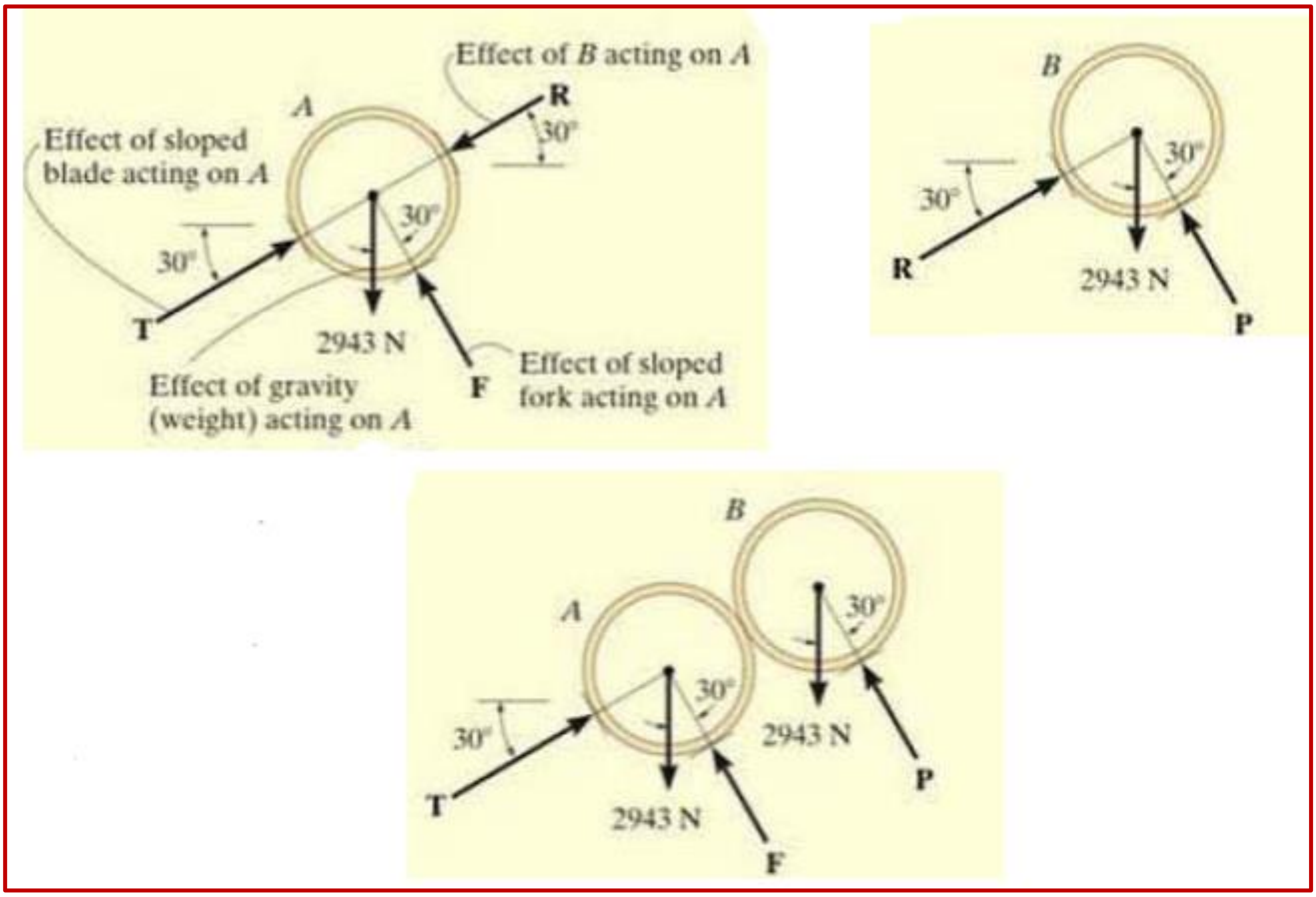
Example - 2

Two smooth pipes, each having a mass of (300 kg), are supported by the forked tines of the tractor. Draw the F,B.D for each pipe and both pipes together.



Solution:

Free Body Diagram (F. B. D)



3.3. General Procedure for the Solution of Problems in Equilibrium

1. Determine the given data and the unknown.
2. Draw the F.B.D for the member on which the unknown forces are acting.
3. Determine the type of force system acting on the F.B.D and the number of independent equations of equilibrium.
4. Compare the number of unknown on the F.B.D with the number of independent equations of equilibrium.
   A. If the number of equations=the number of unknowns, then start the solution.
   B. If the number of unknown > the number of independent equations, then draw F.B.D. for another body and repeat step 3 and 4.
5. If the number of unknowns in the second F.B.D = the number of equations then solve the problem. If it is not repeat step 4-6
6. If there are still too many unknowns after drawing F.B.D for all bodies, then the problem is statically indeterminate.

3.4. Equilibrium of Force System

The body is said to be in equilibrium if the resultant of all forces acting on it is zero. There are two major types of static equilibrium, namely, translational equilibrium and rotational equilibrium.

a) Formulas Concurrent force system

$$\sum F_x = 0\,, \qquad \sum F_y = 0$$

b) Parallel Force System Non-Concurrent

$$\sum F_x = 0\,, \qquad \sum M_O = 0$$

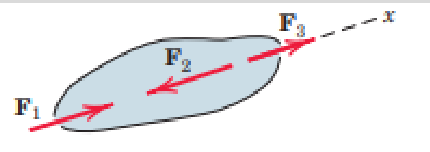
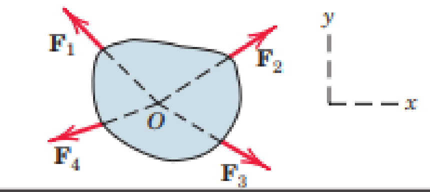
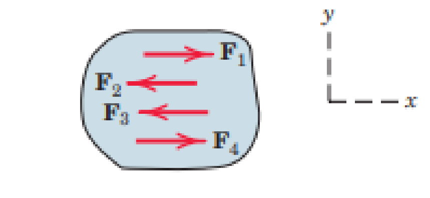
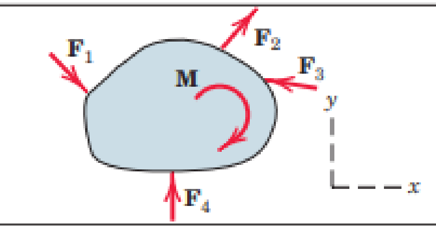
c) Non-Parallel Force System

$$\sum F_x = 0\,, \qquad \sum F_y = 0\,, \qquad \sum M_O = 0$$

3.5. Important Points for Equilibrium Forces

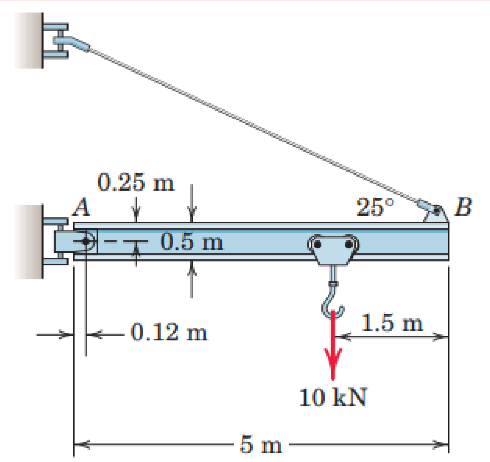
1. Two forces are in equilibrium if they are equal and oppositely directed.
2. Three coplanar forces in equilibrium are concurrent.

3. Three or more concurrent forces in equilibrium form a close polygon when connected in head to-tail manner.

CATEGORIES OF EQUILIBRIUM IN TWO DIMENSIONS		
Force System	Free-Body Diagram	Independent Equations
1. Collinear		$\Sigma F_x = 0$
2. Concurrent at a point		$\Sigma F_x = 0$ $\Sigma F_y = 0$
3. Parallel		$\Sigma F_x = 0$ $\quad \Sigma M_z = 0$
4. General		$\Sigma F_x = 0$ $\quad \Sigma M_z = 0$ $\Sigma F_y = 0$

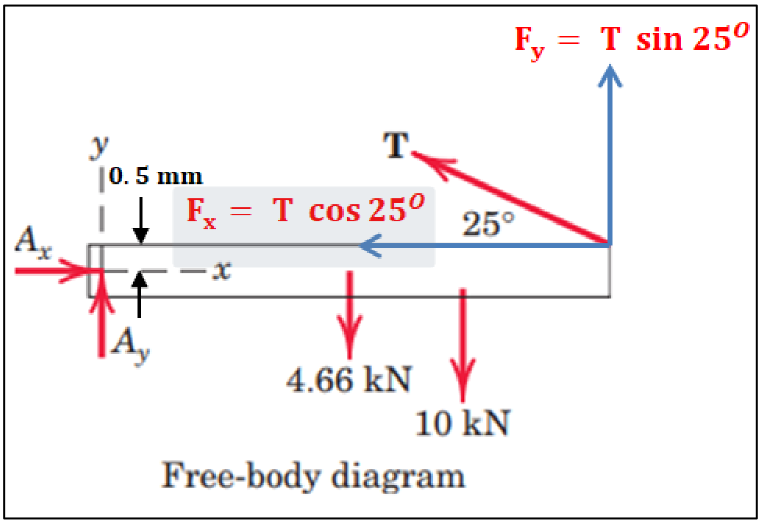
Example – 3

Determine the magnitude T of the tension in the supporting cable and the magnitude of the force on the pin at A for the jib crane shown. The beam AB is a standard (0.5 m) I-beam with a mass of (95 kg) per meter of length?



Solution:

Draw free body diagram (F. B. D)



$$\Sigma F_x = 0 \qquad \Sigma F_y = 0\,, \qquad \Sigma M_A = 0$$

$$\circlearrowright +\Sigma M_A = -(T\cos 25^o)\times 0.25 - (T\sin 25^o)(5-0.12) + 10\times(5-1.5-0.12) + 4.66(2.5-0.12) = 0$$

$$-0.227\,T - 2.062\,T + 33.8 + 11.091 = 0$$

$$-2.289\,T + 44.891 = 0$$

$$T = \frac{44.891}{2.289} = 19.612\ KN$$

$$\Sigma F_x = A_x - T\cos 25^o = 0$$

$$A_x - 19.612\times\cos 25^o = 0$$

$$A_x = 17.775\ KN$$

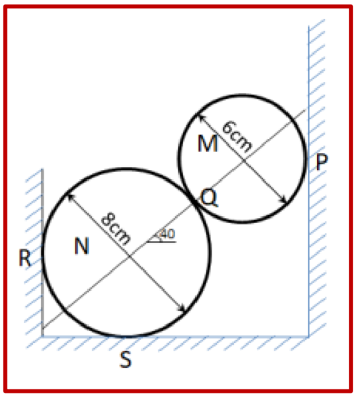
$$\Sigma F_y = A_y + T\sin 25^o - 4.66 - 10 = 0$$

$$A_y = -19.612\times\sin 25^o + 4.66 + 10 = 6.372\ KN$$

$$A = \sqrt{A_x^2 + A_y^2} = \sqrt{17.775^2 + 6.372^2} = 18.883\ KN$$

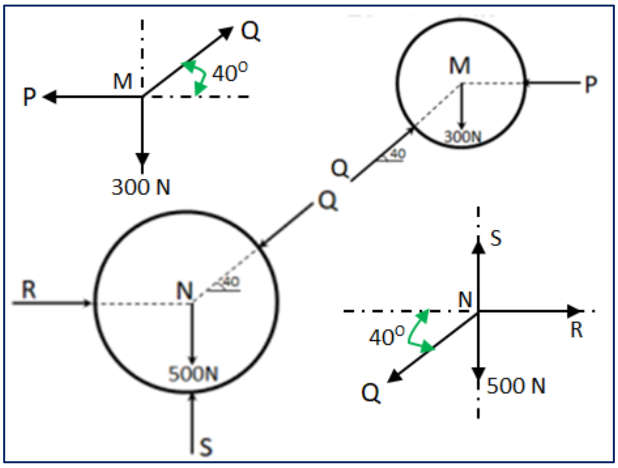
Example – 4

The (300 N) shaft M and the (500 N) shaft N are supported as shown in the figure. Neglecting friction at the contact surfaces P,Q,R and S, determine the reaction at (R and S) on shaft N.?



Solution:

Draw free body diagram (F. B. D)



From F.B.D of M

$$\uparrow \sum F_y = 0$$

$Q \sin 40 - 300 = 0 \qquad \therefore Q = 467N$ on M

From the F.B.D. of N

$$\uparrow \sum F_y = 0$$

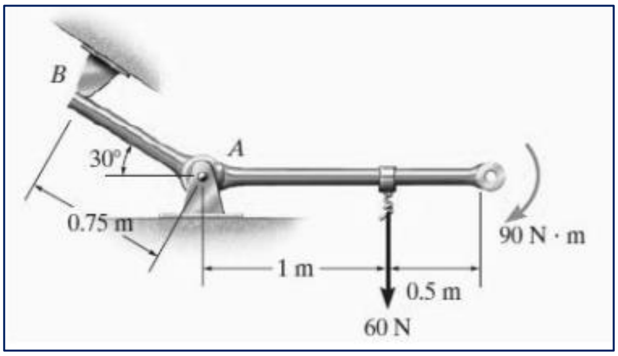
$S - 500 - Q\sin 40 = 0 \qquad \therefore S = 800N \uparrow$ on N

$$\rightarrow \sum F_x = 0$$

$R - Q\cos 40 = 0 \qquad \therefore R = 358N \rightarrow$ on N

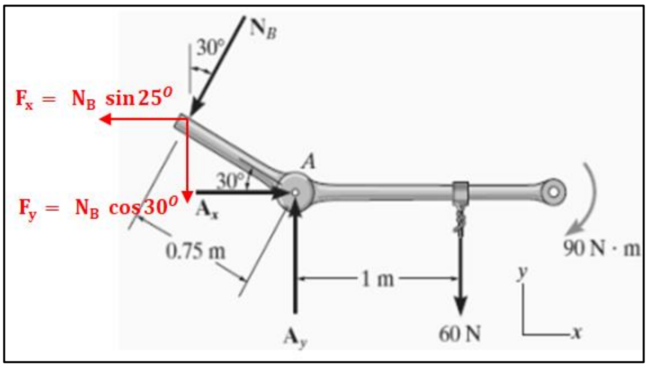
Example – 5

The member shown in Figure is pin-connected at **A** and rests against a smooth support at **B**. Determine the horizontal and vertical components of reaction at the pin **A.**



Solution:

Draw free body diagram (F. B. D)



Equations of Equilibrium: Summing moments about (A), to find direct solution for (N_B).

$$\circlearrowleft +\Sigma M_A = 0$$

$$\circlearrowleft +\Sigma M_A = 90 + 60 \times 1 - N_B \times 0.75$$

$$0 = 90 + 60 - N_B \times 0.75$$

$$N_B = \frac{150}{0.75} = 200\ N$$

$$\Sigma F_x = 0$$

$$\Sigma F_x = A_x - 200 \sin 30^o$$

$$0 = A_x - 200 \sin 30^o$$

$$A_x = 200\, sin\, 30^o = 100\ N$$

$$\Sigma F_y = 0$$

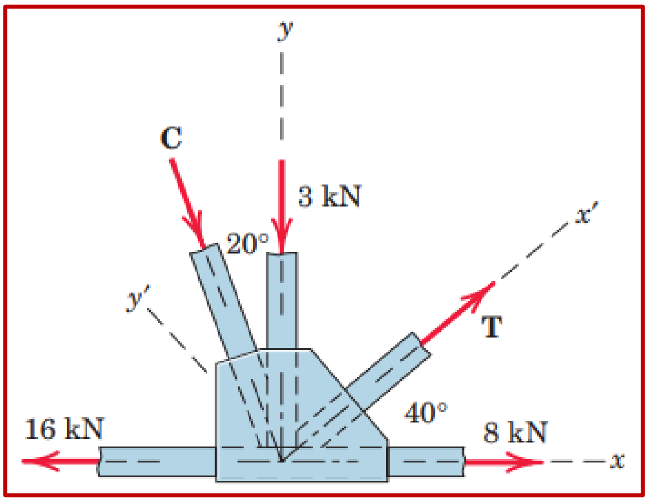
$$\Sigma F_y = A_y - 200 \cos 30^o - 60$$

$$0 = A_y - 200\, cos\, 30^o - 60$$

$$A_y = 200\, cos\, 30^o + 60 = 233\ N$$

3.6. Chapter Questions

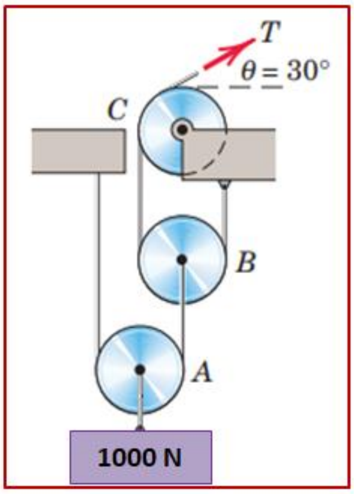
1. Determine the magnitudes of the forces C and T, which, along with the other three forces shown, act on the bridge-truss joint?



{*Answer:* $T = 9.09\ KN$, $C = 3.03\ KN$}

---

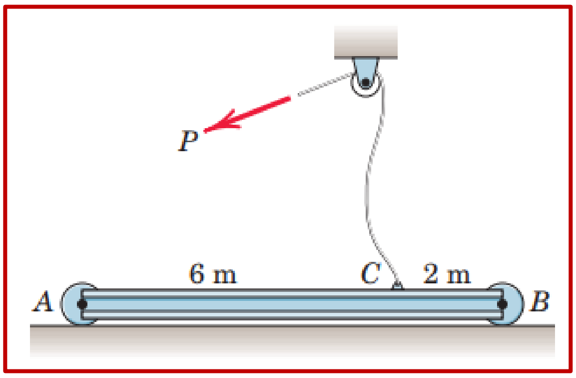
2. Determine the magnitudes of the forces C and T, which, along with the other three forces shown, act on the bridge-truss joint?



{*Answer:* $F = 250\ N$}

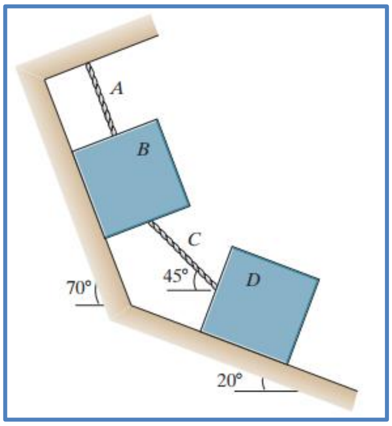
---

3. The uniform (100 kg) I-beam is supported initially by its end rollers on the horizontal surface at A and B. By means of the cable at C it is desired to elevate end B to a position (3 m) above end A. Determine the required tension P, the reaction at A, and the angle made by the beam with the horizontal in the elevated position.



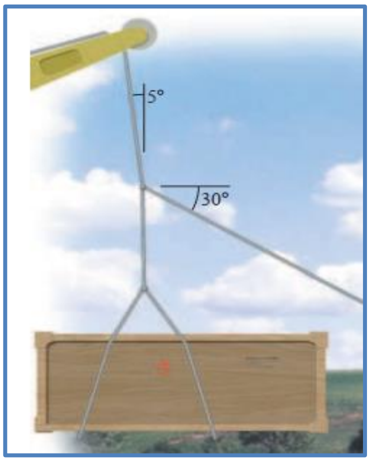
{Answer: $P = 654\ N\ ,\ \theta = 22^o$}

4. Each box weighs 40 N. The angles are measured relative to the horizontal. The surfaces are smooth. Determine the tension in the rope A and the normal force exerted on box B by the inclined surface?



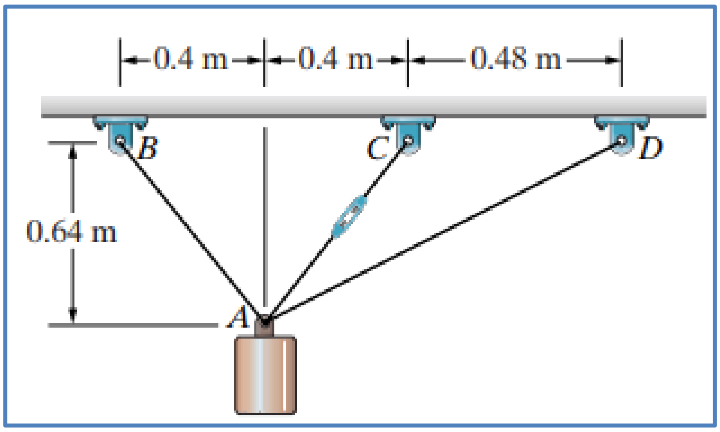
{Answer: $T_A = 51.2\ N\ ,\ N_B = 7.03\ N$}

5. The construction worker exerts a 90 N force on the rope to hold the crate in equilibrium in the position shown. What is the weight of the crate?



{Answer: $W = 935.9\ N$}

6. The 20-kg mass is suspended from three cables. Cable AC is equipped with a turnbuckle so that its tension can be adjusted and a strain gauge that allows its tension to be measured. If the tension in cable AC is 40 N, what are the tensions in cables AB and AD?



{Answer: $T_{AB} = 144.1\ N$, $T_{AD} = 68.2\ N$}

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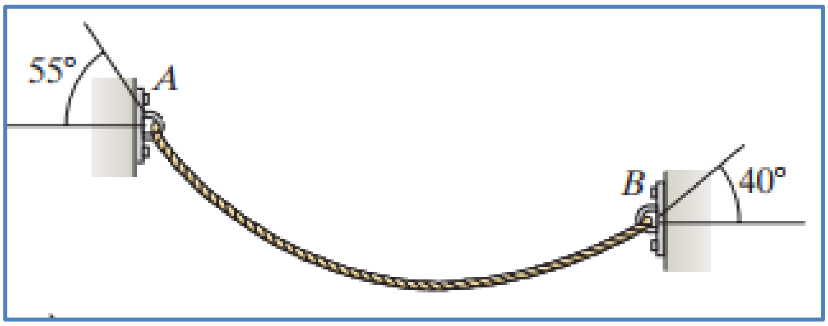
7. A heavy rope used as a mooring line for a cruise ship sags as shown. If the mass of the rope is 90 kg, what are the tensions in the rope at A and B?



{Answer: $T_A = 679\ N$, $T_B = 508\ N$}

Chapter 4

Centroids and Centers of Gravity

4.1. Introduction

A centroid is a weighted average like the center of gravity, but weighted with a geometric property like area or volume, and not a physical property like weight or mass. This means that centroids are properties of pure shapes, not physical objects. They represent the coordinates of the "middle" of the shape.

To design the structure for supporting a water tank, we will need to know the weight of the tank and water as well as the locations where the resultant forces representing these distributed loads act.

4.2. Objectives

1. To discuss the concept of the center of gravity, center of mass, and the centroid.
2. To show how to determine the location of the center of gravity and centroid for a system of discrete particles and a body of arbitrary shape.

4.3. Centroids and Centers of Gravity

A centroid is the geometric center of a geometric object: a one-dimensional curve, a two-dimensional area or a three-dimensional volume. Centroids are useful for many situations in Statics and subsequent courses, including the analysis of distributed forces, beam bending, and shaft torsion.

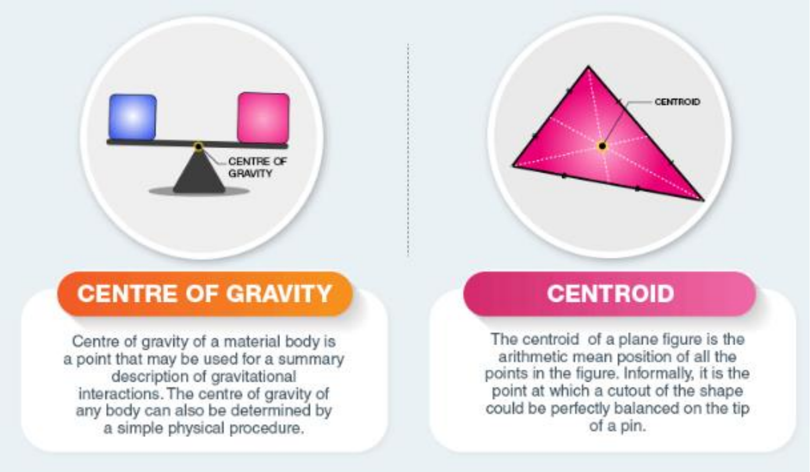
Two related concepts are the **center of gravity**, which is the average location of an object's weight, and the **center of mass** which is the average location of an object's mass. In many engineering situations, the centroid, center of mass, and center of gravity are all coincident. Because of this, these three terms are often used interchangeably without regard to their precise meanings.

Consciously and subconsciously use centroids for many things in life and engineering, including:

1. Keeping your body's balance: Try standing up with your feet together and leaning your head and hips in front of your feet. You have just moved your body's center of gravity out of line with the support of your feet.
2. Computing the stability of objects in motion like cars, airplanes, and boats: By understanding how the center of gravity interacts with the accelerations caused by motion, we can compute safe speeds for sharp curves on a highway.
3. Designing the structural support to balance the structure's own weight and applied loadings on buildings, bridges, and dams: We design most large infrastructure not to move. To keep it from moving, we must understand how the structure's weight, people, vehicles, wind, earth pressure, and water pressure balance with the structural supports.

4.4. Difference between Centre of Gravity and Centroid

1. In an object, a center of mass is referred to as the point where the whole object's mass is focused, which means the point's mass is represented as the whole object's mass. The center of gravity of any object is the point where gravity acts on the body.
2. On the other hand, the centroid is referred to as the geometrical center of a uniform density object. This means the object has its weight distributed equally across all body parts. If the body is homogeneous (having constant density), its center of gravity is equivalent to the centroid.



4.5. Equations for Centroids

The defining equations for centroids are similar to the equations for Centers of Gravity , but with volume used as the weighting factor for three-dimensional shapes.

Centers of Gravity Equations

$$\bar{x} = \frac{\sum \bar{x}_i W_i}{\sum W_i} \quad \bar{y} = \frac{\sum \bar{y}_i W_i}{\sum W_i} \quad \bar{z} = \frac{\sum \bar{z}_i W_i}{\sum W_i}$$

Centroids Equations

$$\bar{x} = \frac{\sum \bar{x}_i V_i}{\sum V_i} \quad \bar{y} = \frac{\sum \bar{y}_i V_i}{\sum V_i} \quad \bar{z} = \frac{\sum \bar{z}_i V_i}{\sum V_i},$$

and area for two-dimensional shapes

$$\bar{x} = \frac{\sum \bar{x}_i A_i}{\sum A_i} \quad \bar{y} = \frac{\sum \bar{y}_i A_i}{\sum A_i}.$$

If the shape has an axis of symmetry, every point on one side of the axis is mirrored by another point equidistant on the other side. One has a positive distance from the axis, and the other is the same distance away in the negative direction. These two points will add to zero the numerator, as will every other point making up the shape, and the first moment will be zero. This means that the centroid must lie along the line of symmetry if there is one. If a shape has multiple symmetry lines, then the centroid must exist at their intersection.

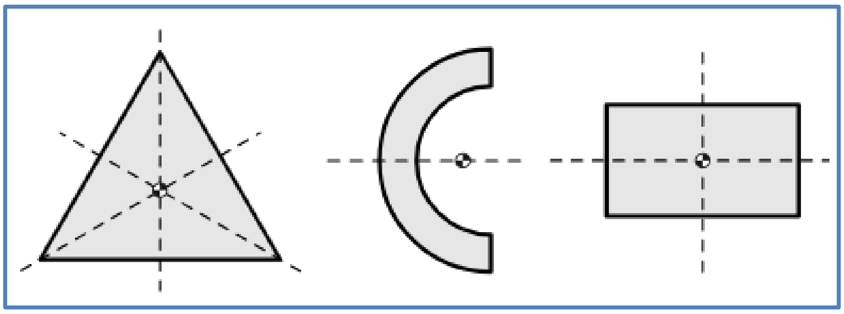


Figure . Centroids lie upon axes of symmetry

Since rectangles, circles, cubes, spheres, etc. have multiple lines of symmetry, their centroids must be exactly in the center as we would expect.

Table. Centroids of Common Shapes

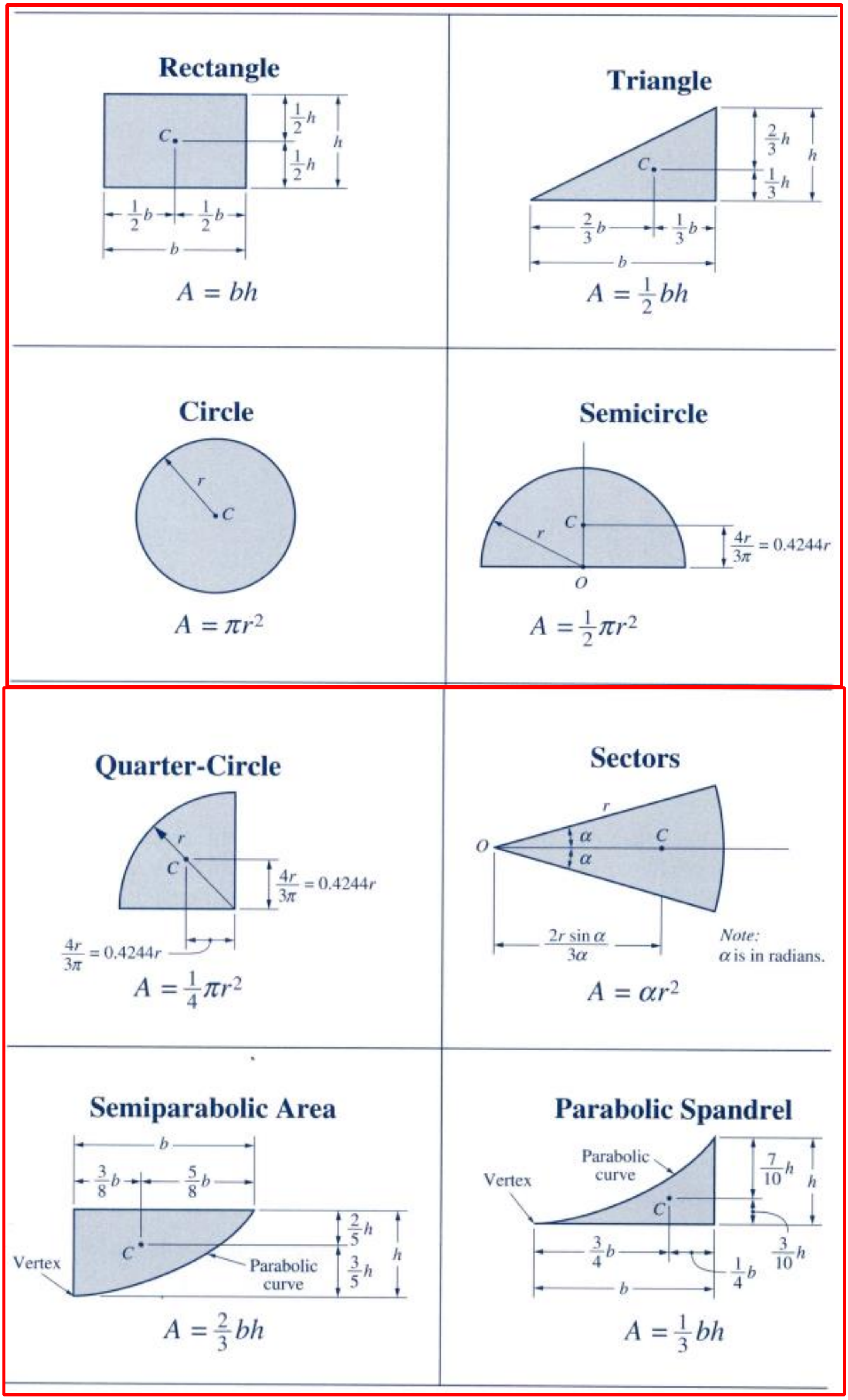
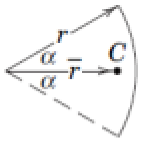
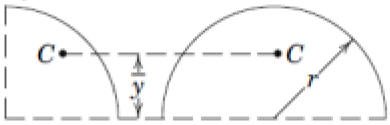
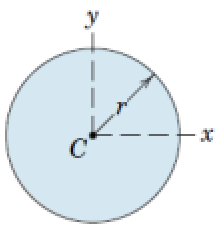
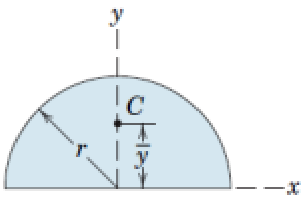
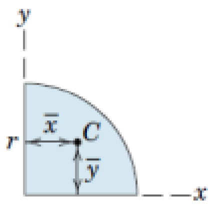
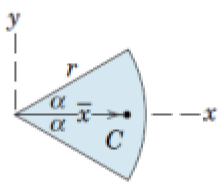
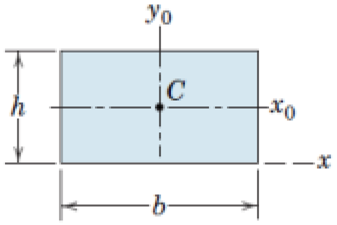
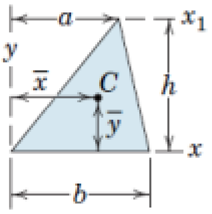
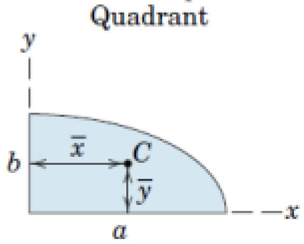
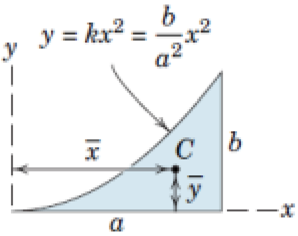
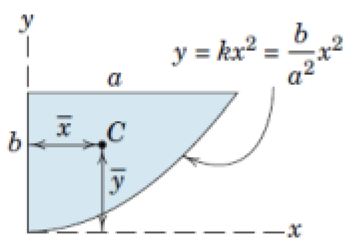


FIGURE	CENTROID	AREA MOMENTS OF INERTIA
Arc Segment 	$\bar{r} = \dfrac{r \sin \alpha}{\alpha}$	—
Quarter and Semicircular Arcs 	$\bar{y} = \dfrac{2r}{\pi}$	—
Circular Area 	—	$I_x = I_y = \dfrac{\pi r^4}{4}$ $I_z = \dfrac{\pi r^4}{2}$
Semicircular Area 	$\bar{y} = \dfrac{4r}{3\pi}$	$I_x = I_y = \dfrac{\pi r^4}{8}$ $\bar{I}_x = \left(\dfrac{\pi}{8} - \dfrac{8}{9\pi}\right) r^4$ $I_z = \dfrac{\pi r^4}{4}$
Quarter-Circular Area 	$\bar{x} = \bar{y} = \dfrac{4r}{3\pi}$	$I_x = I_y = \dfrac{\pi r^4}{16}$ $\bar{I}_x = \bar{I}_y = \left(\dfrac{\pi}{16} - \dfrac{4}{9\pi}\right) r^4$ $I_z = \dfrac{\pi r^4}{8}$
Area of Circular Sector 	$\bar{x} = \dfrac{2}{3} \dfrac{r \sin \alpha}{\alpha}$	$I_x = \dfrac{r^4}{4}(\alpha - \dfrac{1}{2}\sin 2\alpha)$ $I_y = \dfrac{r^4}{4}(\alpha + \dfrac{1}{2}\sin 2\alpha)$ $I_z = \dfrac{1}{2} r^4 \alpha$

Rectangular Area 	—	$I_x = \frac{bh^3}{3}$ $\bar{I}_x = \frac{bh^3}{12}$ $\bar{I}_z = \frac{bh}{12}(b^2 + h^2)$
Triangular Area 	$\bar{x} = \frac{a+b}{3}$ $\bar{y} = \frac{h}{3}$	$I_x = \frac{bh^3}{12}$ $\bar{I}_x = \frac{bh^3}{36}$ $I_{x_1} = \frac{bh^3}{4}$
Area of Elliptical Quadrant 	$\bar{x} = \frac{4a}{3\pi}$ $\bar{y} = \frac{4b}{3\pi}$	$I_x = \frac{\pi ab^3}{16}$, $\bar{I}_x = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right)ab^3$ $I_y = \frac{\pi a^3 b}{16}$, $\bar{I}_y = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right)a^3 b$ $I_z = \frac{\pi ab}{16}(a^2 + b^2)$
Subparabolic Area Area $A = \frac{ab}{3}$ 	$\bar{x} = \frac{3a}{4}$ $\bar{y} = \frac{3b}{10}$	$I_x = \frac{ab^3}{21}$ $I_y = \frac{a^3 b}{5}$ $I_z = ab\left(\frac{a^2}{5} + \frac{b^2}{21}\right)$
Parabolic Area Area $A = \frac{2ab}{3}$ 	$\bar{x} = \frac{3a}{8}$ $\bar{y} = \frac{3b}{5}$	$I_x = \frac{2ab^3}{7}$ $I_y = \frac{2a^3 b}{15}$ $I_z = 2ab\left(\frac{a^2}{15} + \frac{b^2}{7}\right)$

4.6. Steps for Analysis

1. Divide the body into pieces that are known shapes. Holes are considered as pieces with negative weight or size.
2. Make a table with the first column for segment number, the second column for weight, mass, or size (depending on the problem), the next set of columns for the moment arms, and, finally, several columns for recording results of simple intermediate calculations.
3. Fix the coordinate axes, determine the coordinates of the center of gravity of centroid of each piece, and then fill in the table.
4. Sum the columns to get x, y, and z. Use formulas like.

$$x_C = \frac{\Sigma\, x_i\,.L_i}{\Sigma\, L_i}\;, \qquad y_C = \frac{\Sigma\, y_i\,.L_i}{\Sigma\, L_i}\;, \qquad z_C = \frac{\Sigma\, z_i\,.L_i}{\Sigma\, L_i}$$

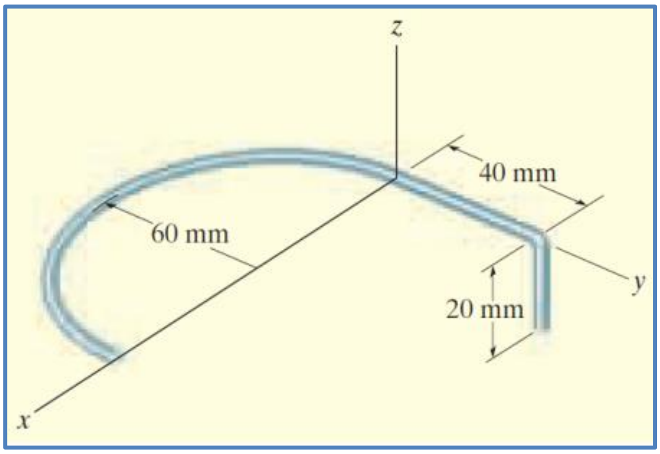
$$x_C = \frac{\Sigma\, x_i\,.A_i}{\Sigma\, A_i}\;, \qquad y_C = \frac{\Sigma\, y_i\,.A_i}{\Sigma\, A_i}\;, \qquad z_C = \frac{\Sigma\, z_i\,.A_i}{\Sigma\, A_i}$$

$$x_C = \frac{\Sigma\, x_i\,.V_i}{\Sigma\, V_i}\;, \qquad y_C = \frac{\Sigma\, y_i\,.V_i}{\Sigma\, V_i}\;, \qquad z_C = \frac{\Sigma\, z_i\,.V_i}{\Sigma\, V_i}$$

$$x_C = \frac{\Sigma\, x_i\,.m_i}{\Sigma\, m_i}\;, \qquad y_C = \frac{\Sigma\, y_i\,.m_i}{\Sigma\, m_i}\;, \qquad z_C = \frac{\Sigma\, z_i\,.m_i}{\Sigma\, m_i}$$

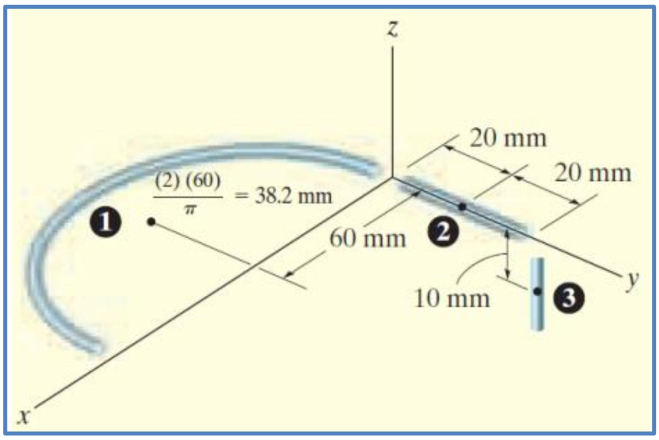
Example - 1

Locate the centroid of the wire shown in the figure below?



Solution:

1. The wire is divided into three segments as shown in the figure below.



2. Moment Arms. The location of the centroid for each segment is determined and indicated in the figure. In particular, the centroid of segment (1) is determined either by integration or by using the table .

Seg.	$L_i\ (m^2)$	$x_i\ (m)$	$x_i\,.L_i\ (m^3)$	$y_i\ (m)$	$y_i\,.L_i\ (m^3)$	$z_i\ (m)$	$z_i\,.L_i\ (m^3)$
1	$\pi(60) = 188.5$	60	11310	-38.2	-7200	0	0
2	40	0	0	20	800	0	0
3	20	0	0	40	800	-10	-200
Sum	**248.5**		**11310**		**-5600**		**-200**

$$x_C = \frac{\Sigma\, x_i\,.L_i}{\Sigma\, L_i} = \frac{11310}{248.5} = 45.5\ mm$$

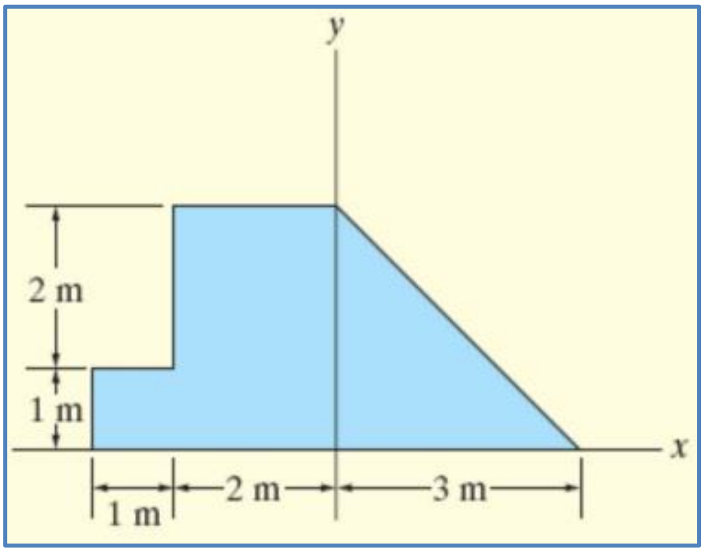
$$y_C = \frac{\Sigma\, y_i\,.L_i}{\Sigma\, L_i} = \frac{-5600}{248.5} = -22.5\ mm$$

$$z_C = \frac{\Sigma\, z_i\,.L_i}{\Sigma\, L_i} = \frac{-200}{248.5} = -0.805\ mm$$

$$C = (45.5, -22.5, -0.805)$$

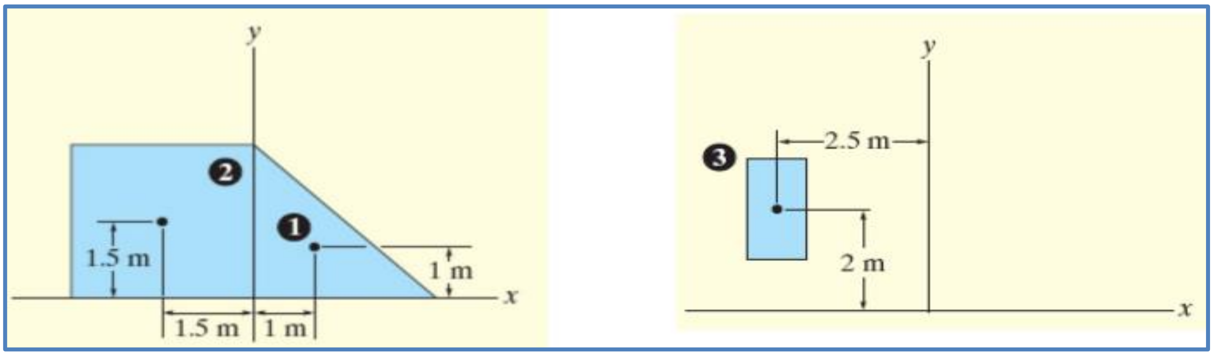
Example - 2

Locate the centroid of the plate area?



Solution:

Plate divided into 3 segments. Area of small rectangle considered "negative".



Solution Moment

Arm Location of the centroid for each piece is determined and indicated in the diagram.

Segment	$A_i\ (m^2)$	$x_i\ (m)$	$x_i\,.A_i\ (m^3)$	$y_i\ (m)$	$y_i\,.A_i\ (m^3)$
1	$\frac{1}{2}(3)(3) = 4.5$	1	4.5	1	4.5
2	$(3)(3) = 9$	− 1.5	− 13.5	1.5	13.5
3	$-(2)(1) = -2$	− 2.5	5	2	− 4
Sum	**11.5**		**− 4**		**14**

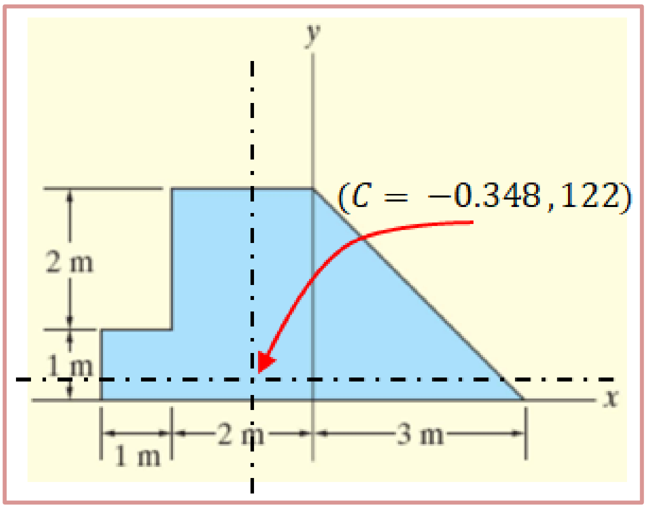
Summations

$$x_C = \frac{\Sigma\, x_i\,.A_i}{\Sigma\, A_i} = \frac{-4}{11.5} = -0.348\, mm$$

$$y_C = \frac{\Sigma\, y_i\,.A_i}{\Sigma\, A_i} = \frac{14}{11.5} = 1.22\, mm$$

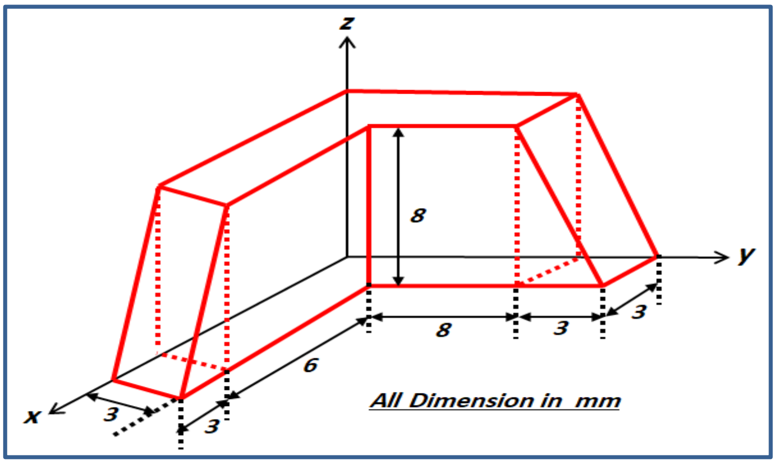
$$z_C = \frac{\Sigma\, z_i\,.A_i}{\Sigma\, A_i} = 0$$

$$C = (-0.348, 1.22)$$



Example - 3

Locate the center of mass of the bracket and shaft combination. The vertical face is made from sheet metal which has a mass of 25 kg/m^2. The material of the horizontal base has a mass of 40 kg/m^2, and the steel shaft has a density of 7830 Kg/m^3. (All dimensions in the figure are in millimeters)?



Solution

Seg.	$V_i\ (mm^2)$	$x_i\ (mm)$	$x_i\,.V_i\ (mm^3)$	$y_i\ (mm)$	$y_i\,.V_i\ (mm^3)$	$z_i\ (mm)$	$z_i\,.V_i\ (mm^3)$
1	36	10	360	1.5	54	$\frac{8}{3}$	96
2	216	4.5	972	1.5	324	4	864
3	192	1.5	288	7	1344	4	768
4	36	1.5	54	12	432	$\frac{8}{3}$	96
Sum	**480**		**1674**		**2154**		**1824**

$$x_C = \frac{\Sigma\, x_i\,.V_i}{\Sigma\, V_i} = \frac{1674}{480} = 3.49\ mm$$

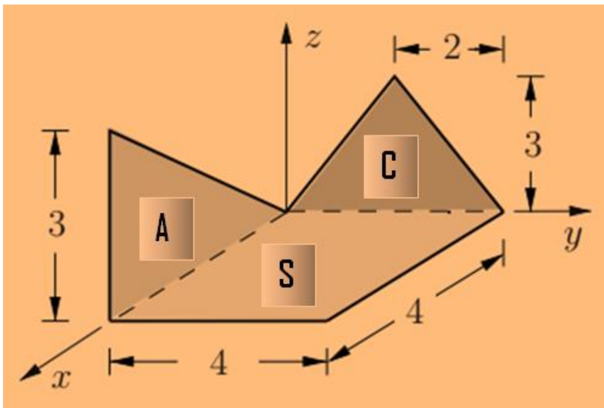
$$y_C = \frac{\Sigma\, y_i\,.V_i}{\Sigma\, V_i} = \frac{2154}{480} = 4.49\ mm$$

$$z_C = \frac{\Sigma\, z_i\,.V_i}{\Sigma\, V_i} = \frac{1824}{480} = 3.80\ mm$$

$$C = (3.49, 4.49, 3.80)\ mm$$

Example - 4

A thin sheet, copper and aluminum with thicknesses ($t_S = 0.03\,m$, $t_C = 0.02\,m, t_A = 0.04\,m$ and densities, ($\rho_S = 7850\,Kg/m^3$, $\rho_C = 8960\,Kg/m^3, \rho_A = 2700\,Kg/m^3$, consisting of a square and two triangles, is bent to the depicted figure (measurements in meter). Locate the center of gravity?



Solution

The body is composed by three parts with already known location of centers of mass. The location of the center of mass of the complete system can be determined from:

$$x_C = \frac{\Sigma\, x_i\,.m_i}{\Sigma\, m_i}\,, \qquad y_C = \frac{\Sigma\, y_i\,.m_i}{\Sigma\, m_i}\,, \qquad z_C = \frac{\Sigma\, z_i\,.m_i}{\Sigma\, m_i}$$

$$x_C = \frac{\Sigma\rho_i\,.x_i\,.V_i}{\Sigma\rho_i\,.V_i}\,, \qquad y_C = \frac{\Sigma\rho_i\,.y_i\,.V_i}{\Sigma\rho_i\,.V_i}\,, z_C = \frac{\Sigma\rho_i\,.z_i\,.V_i}{\Sigma\rho_i\,.V_i}$$

The total area is:

$$V_S = 4 \times 4 \times 0.03 = 0.48\,m^3$$

$$V_C = \frac{1}{2} \times 4 \times 3 \times 0.02 = 0.12\,m^3$$

$$V_A = \frac{1}{2} \times 4 \times 3 \times 0.04 = 0.24\,m^3$$

$$\Sigma V_i = 0.48 + 0.12 + 0.24 = 0.84\,m^3$$

$$\Sigma\rho_i\,.V_i = 7850 \times 0.48 + 8960 \times 0.12 + 2700 \times 0.24 = 0.48 + 0.12 + 0.24$$
$$= 5491\,Kg$$

$$x_C = \frac{\Sigma\rho_i\,.x_i\,.V_i}{\Sigma\rho_i\,.V_i}$$

$$y_C = \frac{\Sigma\rho_i \,.\, y_i \,.\, V_i}{\Sigma\rho_i \,.\, V_i}$$

$$z_C = \frac{\Sigma\rho_i \,.\, z_i \,.\, V_i}{\Sigma\rho_i \,.\, V_i}$$

Calculating the first area moments of the total system about each axis, in each case one first moment of a subsystem drops out because of zero distance: $x_C = 0,\ y_A = 0, and\ z_S = 0$. Thus, we obtain:

$$x_C = \frac{x_s.m_s + x_A.m_A}{\Sigma m_i} = \frac{2 \times 3768 + \left(\frac{2}{3} \times 4\right) \times 648}{5491} = 1.68\, m\,,$$

$$y_C = \frac{x_s.m_s\,. + x_c.m_c\,.}{\Sigma m_i} = \frac{2 \times 3768 + 2 \times 1075}{5491} = 1.76\, m$$

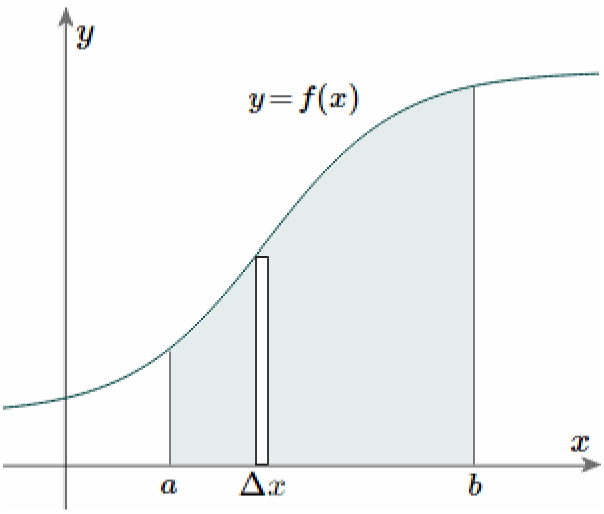
$$z_C = \frac{x_C.m_C + x_A.m_A}{\Sigma m_i} = \frac{1 \times 1075 + 1 \times 648}{5491} = 0.31 m$$

$$C = (1.68, 1.76, 0.31)\ m$$

4.7. Finding the Centroid via the First Moment Integral

Centroid for Curved Areas

Taking the simple case first, we aim to find the centroid for the area defined by a function $f(x)$, and the vertical lines $x = a$ and $x = b$ as indicated in the following figure.



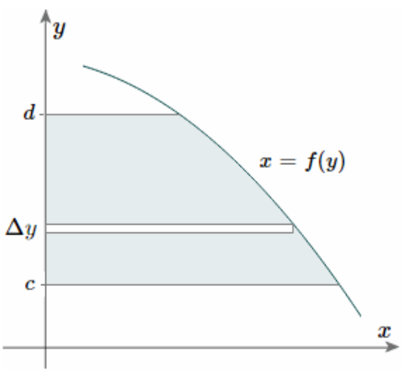
To find the centroid, we use the same basic idea that we were using for the straight-sided case above. The "typical" rectangle indicated is x units from the y-axis, and it has width Δx (which becomes dx when we integrate) and height $y = f(x)$.

Generalizing from the above rectangular areas case, we multiply these 3 values (x, $f(x)$ and Δx, which will give us the area of each thin rectangle times its distance from the x-axis), then add them. If we do this for infinitesimally small strips, we get the x-coordinates of the centroid using the total moments in the x-direction, given by:

$$\overline{x} = \frac{\text{total moments}}{\text{total area}} = \frac{1}{A}\int_a^b x\ f(x)\ dx$$

And, considering the moments in the y-direction about the x-axis and re-expressing the function in terms of y, we have:

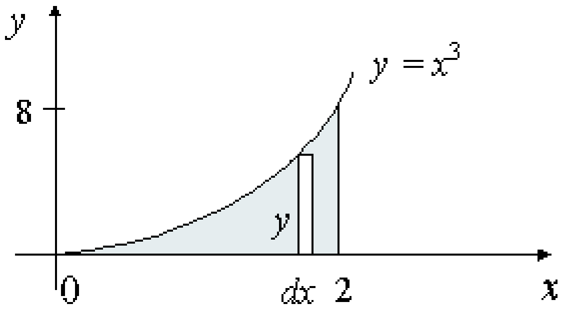
$$\overline{y} = \frac{\text{total moments}}{\text{total area}} = \frac{1}{A}\int_c^d y\ f(y)\ dy$$



Example - 5

Find the centroid of the area bounded by $y = x^3$, $x = 2$ and the x-axis.

Here is the area under consideration:



In this case, $y = f(x) = x^3, a = 0, b = 2.$

We find the shaded area first:

$$A = \int_0^2 x^3\,dx = \left[\frac{x^4}{4}\right]_0^2 = \frac{16}{4} = 4$$

Next, using the formula for the x-coordinate of the centroid we have:

$$\begin{aligned}\overline{x} &= \frac{1}{A}\int_a^b x f(x)\,dx \\ &= \frac{1}{4}\int_0^2 x\left(x^3\right)dx \\ &= \frac{1}{4}\int_0^2 \left(x^4\right)dx \\ &= \frac{1}{4}\left[\frac{x^5}{5}\right]_0^2 \\ &= \frac{32}{20} \\ &= 1.6\end{aligned}$$

$$\overline{y} = \frac{1}{A}\int_c^d y(x_2 - x_1)\,dy$$

$$= \frac{1}{4}\int_0^8 y\left(2 - y^{\frac{1}{3}}\right)dy$$

$$= \frac{1}{4}\int_0^8 \left(2y - y^{\frac{4}{3}}\right)dy$$

$$= \frac{1}{4}\left[y^2 - \frac{3y^{\frac{7}{3}}}{7}\right]_0^8$$

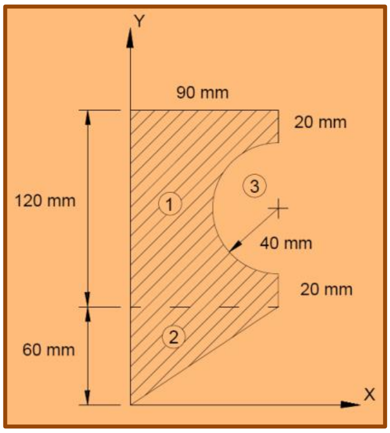
$$= \frac{1}{4}\left[64 - \frac{3 \times 128}{7}\right]$$

$$= 2.29$$

So the centroid for the shaded area is at (1.6, 2.29).

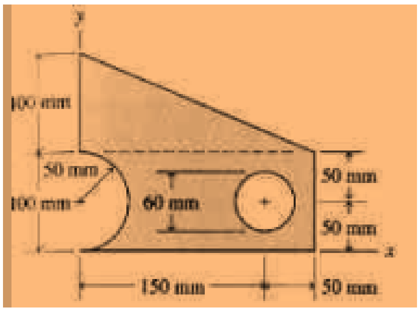
4.8. Chapter Questions

Q1: Locate the centroid of the area shown in the figure below?



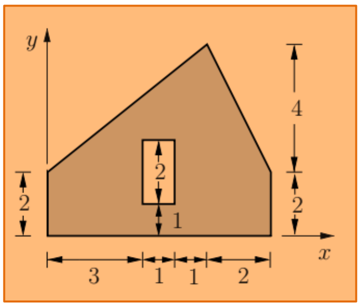
{Answer: $x_C = 34.9\ mm$, and $y_C = 100.4\ mm$}

Q2: Locate the centroid of the area shown in the figure below?



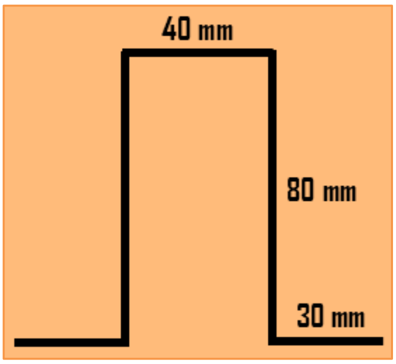
{Answer: $x_C = 92.9\ mm$, and $y_C = 85.8\ mm$}

Q3: Locate the centroid of the depicted area with a rectangular cutout. The measurements are given in meter?



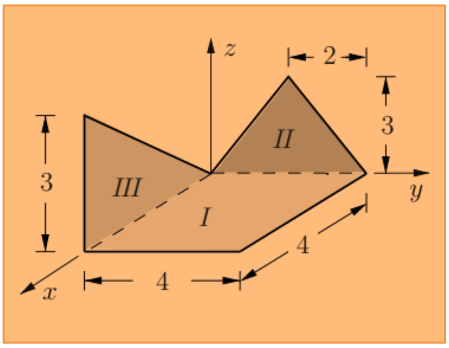
{Answer: $x_C = 3.77\ m$, and $y_C = 2.18\ m$ }

Q_4: A wire with constant thickness is deformed into the depicted figure. The measurements are given in mm. Find the Locate of the centroid?



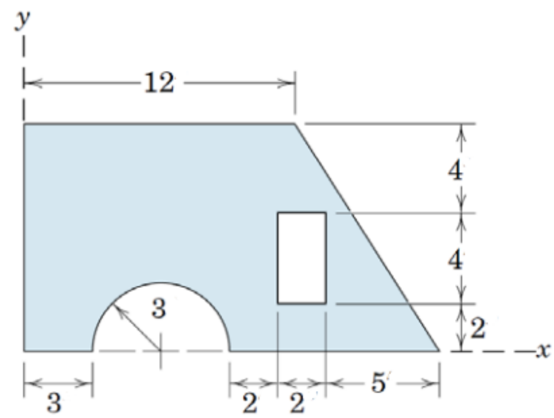
{*Answer:* $y_C = 5.08\ mm$ }

Q_5: A thin sheet with constant thickness and density, consisting of a square and two triangles, is bent to the depicted figure (measurements in meter). Locate the center of gravity?



$x_C = 1.71\ m, \qquad y_C = 1.57, and\ z_C = 0.43\ m$ }

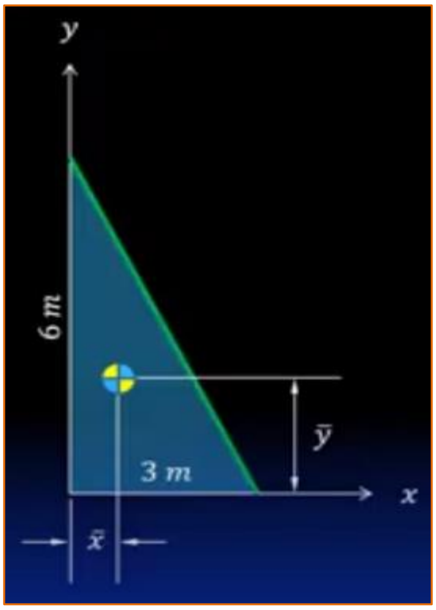
Q_6: Locate the centroid of the area shown in the figure below?



{***Answer:*** $x_C = 7.5\ m\ \ , and\ y_C = 5.08\ m$ }

Q_7: Find the Locate of the centroid of the area shown in the figure below, by using integration?

$$y = 2(3 - x)$$

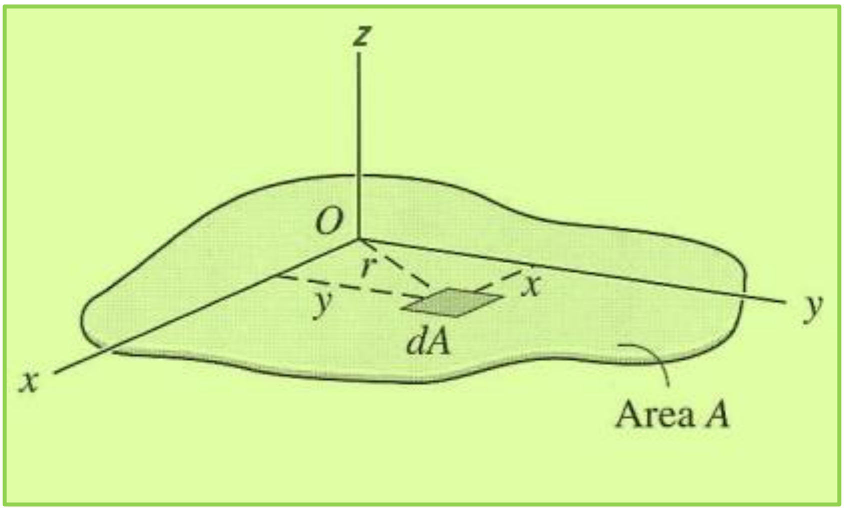


Chapter 5

Moments of Inertia

5.1. Moment of Inertia and Properties of Plane Areas

The Moment of Inertia (I) is a term used to describe the capacity of a cross-section to resist bending. It is always considered with respect to a reference axis such as (X-X) or (Y-Y). It is a mathematical property of a section concerned with a surface area and how that area is distributed about the reference axis (axis of interest). The reference axis is usually a centroid axis. The moment of inertia is also known as the Second Moment of the Area and is expressed mathematically as:



$$I_x = \int_A y^2 dA$$

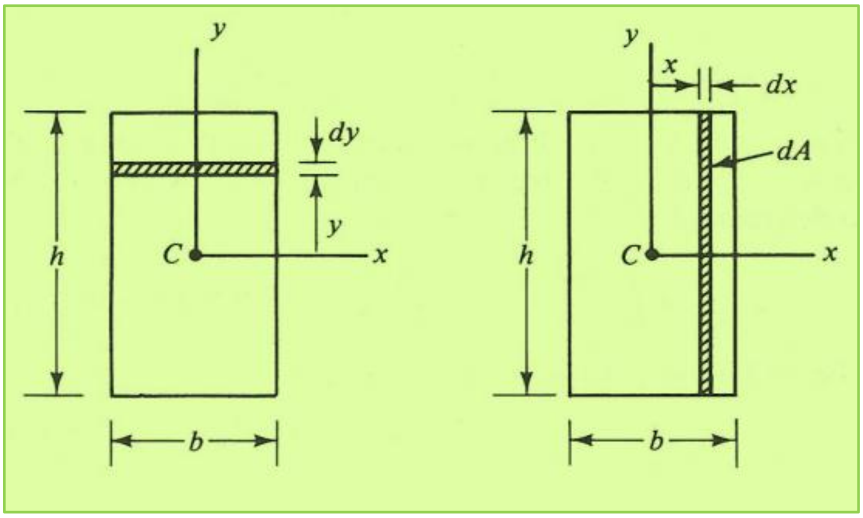
$$I_y = \int_A x^2 dA$$

Where:

y = distance from the x axis to area *dA*

x = distance from the y axis to area d*A*

Example



5.2. Application of moment of inertia

The crank on the oil-pump rig undergoes rotation about a fixed axis that is not at its mass center. The crank develops a kinetic energy directly related to its mass moment of inertia. As the crank rotates, its kinetic energy is converted to potential energy and vice versa.

Is the mass moment of inertia of the crank about its axis of rotation smaller or larger than its moment of inertia about its center of mass.

5.3. Radius of Gyration

The radius of gyration of an area with respect to a particular axis is the square root of the quotient of the moment of inertia divided by the area. It is the distance at which the entire area must be assumed to be concentrated in order that the product of the area and the square of this distance will equal the moment of inertia of the actual area about the given axis. In other words, the radius of gyration describes the way in which the total cross-sectional area is distributed around its centroidal axis. If more area is distributed further from the axis, it will have greater resistance to buckling. The most efficient column section to resist buckling is a circular pipe, because it has its area distributed as far away as possible from the centroid. Rearranging we have:

$$\boldsymbol{I_x = k_x^2 \,.\, A} \quad , \qquad \boldsymbol{I_y = k_y^2 \,.\, A} \quad , \qquad \boldsymbol{I_z = k_z^2 \,.\, A}$$

$$\boldsymbol{k_x = \sqrt{\frac{I_x}{A}}}$$

$$\boldsymbol{k_y = \sqrt{\frac{I_y}{A}}}$$

$$\boldsymbol{k_z = \sqrt{\frac{I_y}{A}}}$$

The radius of gyration is the distance k away from the axis that all the area can be concentrated to result in the same moment of inertia.

5.4. Polar Moment of Inertia

$$I_p = \int_A \rho^2 dA$$
$$I_p = \int_A (x^2 + y^2) dA$$
$$I_p = \int_A x^2 dA + \int_A y^2 dA$$
$$I_p = I_x + I_y$$

In many texts, the symbol J will be used to denote the polar moment of inertia.

$$J = I_x + I_y$$

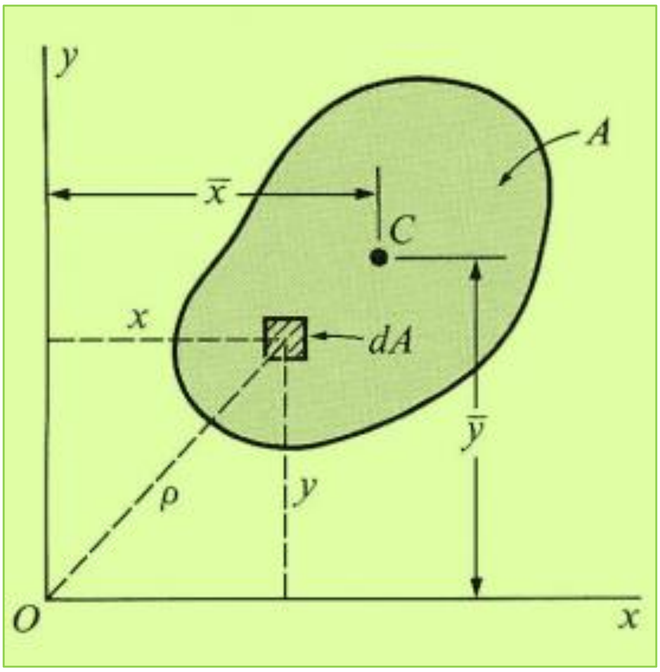
Shear stress formula:

$$\tau = \frac{T_r}{J}$$

5.5. Product of Inertia

$$I_{xy} = \int_A xy\,dA$$

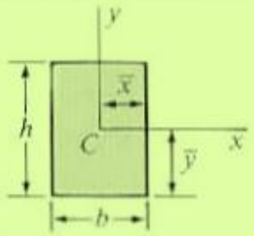
Consider the following:



If an area has at least one axis of symmetry, the product of inertia is zero.

5.6. Properties of Plane Areas

1



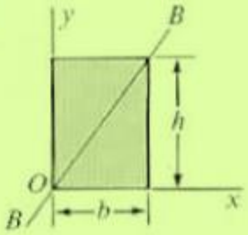
Rectangle (Origin of axes at centroid.)

$$A = bh \qquad \bar{x} = \frac{b}{2} \qquad \bar{y} = \frac{h}{2}$$

$$I_x = \frac{bh^3}{12} \qquad I_y = \frac{hb^3}{12}$$

$$I_{xy} = 0 \qquad I_p = \frac{bh}{12}(h^2 + b^2)$$

2

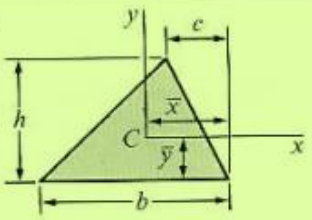


Rectangle (Origin of axes at corner.)

$$I_x = \frac{bh^3}{3} \qquad I_y = \frac{hb^3}{3}$$

$$I_{xy} = \frac{b^2h^2}{4} \qquad I_p = \frac{bh}{3}(h^2 + b^2) \qquad I_{BB} = \frac{b^3h^3}{6(b^2 + h^2)}$$

3



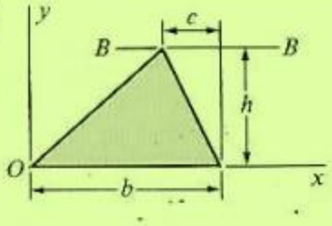
Triangle (Origin of axes at centroid.)

$$A = \frac{bh}{2} \qquad \bar{x} = \frac{b + c}{3} \qquad \bar{y} = \frac{h}{3}$$

$$I_x = \frac{bh^3}{36} \qquad I_y = \frac{bh}{36}(b^2 - bc + c^2)$$

$$I_{xy} = \frac{bh^2}{72}(b - 2c) \qquad I_p = \frac{bh}{36}(h^2 + b^2 - bc + c^2)$$

4

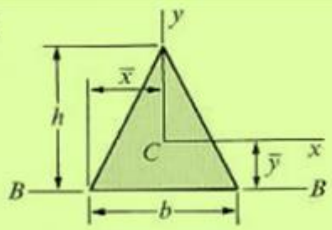


Triangle (Origin of axes at vertex.)

$$I_x = \frac{bh^3}{12} \qquad I_y = \frac{bh}{12}(3b^2 - 3bc + c^2)$$

$$I_{xy} = \frac{bh^2}{24}(3b - 2c) \qquad I_{BB} = \frac{bh^3}{4}$$

5



Isosceles triangle (Origin of axes at centroid.)

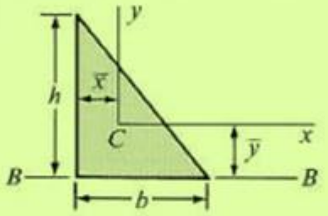
$$A = \frac{bh}{2} \qquad \bar{x} = \frac{b}{2} \qquad \bar{y} = \frac{h}{3}$$

$$I_x = \frac{bh^3}{36} \qquad I_y = \frac{hb^3}{48} \qquad I_{xy} = 0$$

$$I_p = \frac{bh}{144}(4h^2 + 3b^2) \qquad I_{BB} = \frac{bh^3}{12}$$

(Note: For an equilateral triangle, $h = \sqrt{3}b/2$.)

6



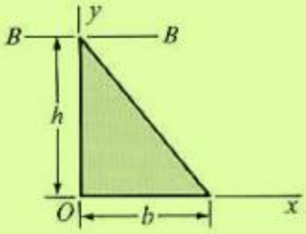
Right triangle (Origin of axes at centroid.)

$$A = \frac{bh}{2} \qquad \bar{x} = \frac{b}{3} \qquad \bar{y} = \frac{h}{3}$$

$$I_x = \frac{bh^3}{36} \qquad I_y = \frac{hb^3}{36} \qquad I_{xy} = -\frac{b^2h^2}{72}$$

$$I_p = \frac{bh}{36}(h^2 + b^2) \qquad I_{BB} = \frac{bh^3}{12}$$

7

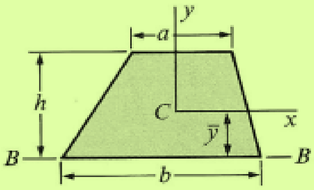


Right triangle (Origin of axes at vertex.)

$$I_x = \frac{bh^3}{12} \qquad I_y = \frac{hb^3}{12} \qquad I_{xy} = \frac{b^2h^2}{24}$$

$$I_p = \frac{bh}{12}(h^2 + b^2) \qquad I_{BB} = \frac{bh^3}{4}$$

8

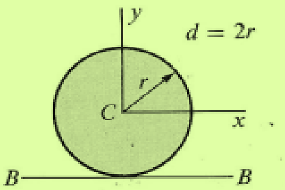


Trapezoid (Origin of axes at centroid.)

$$A = \frac{h(a+b)}{2} \qquad \bar{y} = \frac{h(2a+b)}{3(a+b)}$$

$$I_x = \frac{h^3(a^2 + 4ab + b^2)}{36(a+b)} \qquad I_{BB} = \frac{h^3(3a+b)}{12}$$

9

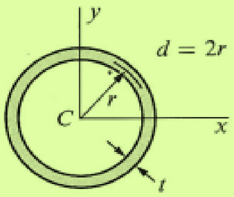


Circle (Origin of axes at center.)

$$A = \pi r^2 = \frac{\pi d^2}{4} \qquad I_x = I_y = \frac{\pi r^4}{4} = \frac{\pi d^4}{64}$$

$$I_{xy} = 0 \qquad I_p = \frac{\pi r^4}{2} = \frac{\pi d^4}{32} \qquad I_{BB} = \frac{5\pi r^4}{4} = \frac{5\pi d^4}{64}$$

10

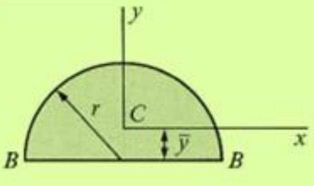


Circular ring (Origin of axes at center.)
Approximate formulas for case when t is small.

$$A = 2\pi rt = \pi dt \qquad I_x = I_y = \pi r^3 t = \frac{\pi d^3 t}{8}$$

$$I_{xy} = 0 \qquad I_p = 2\pi r^3 t = \frac{\pi d^3 t}{4}$$

11

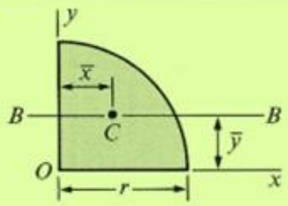


Semicircle (Origin of axes at centroid.)

$$A = \frac{\pi r^2}{2} \qquad \bar{y} = \frac{4r}{3\pi}$$

$$I_x = \frac{(9\pi^2 - 64)r^4}{72\pi} \approx 0.1098r^4 \qquad I_y = \frac{\pi r^4}{8}$$

$$I_{xy} = 0 \qquad I_{BB} = \frac{\pi r^4}{8}$$

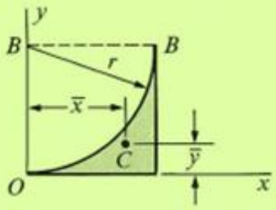
12 

Quarter circle (Origin of axes at center of circle.)

$A = \frac{\pi r^2}{4} \qquad \bar{x} = \bar{y} = \frac{4r}{3\pi}$

$I_x = I_y = \frac{\pi r^4}{16} \qquad I_{xy} = \frac{r^4}{8}$

$I_{BB} = \frac{(9\pi^2 - 64)r^4}{144\pi} \approx 0.05488r^4$

13 

Quarter-circular spandrel (Origin of axes at vertex.)

$A = \left(1 - \frac{\pi}{4}\right) r^2$

$\bar{x} = \frac{2r}{3(4-\pi)} \approx 0.7766r \qquad \bar{y} = \frac{(10 - 3\pi)r}{3(4-\pi)} \approx 0.2234r$

$I_x = \left(1 - \frac{5\pi}{16}\right) r^4 \approx 0.01825r^4 \qquad I_y = I_{BB} = \left(\frac{1}{3} - \frac{\pi}{16}\right) r^4 \approx 0.1370r^4$

5.7. Moment of Inertia of Composite Areas

In the context of calculating the moment of inertia, a composite area is an area consisting of several non-overlapping (disjoint) sub-areas. The boundaries specifying the sub-areas can be explicitly declared by the geometry or arbitrarily chosen. Considering an area as a composite area is to simplify the calculation of the moment of inertia of the whole area. having *simple* shapes with already known or given formulations of moments of inertia. To find the moment of inertia, the following table and equations are applied:

Part	A (mm)	dx (mm)	dy (mm)	A , dx^2 (mm^4)	A , dy^2 (mm^4)	Ix0 (mm^4)	Iy0 (mm^4)
1							
2							
3							
Total				$\Sigma A \cdot d_x^2$	$\Sigma A \cdot d_y^2$	ΣI_{x0}	ΣI_{y0}

$$I_x = \Sigma I_{x0} + \Sigma A \cdot d_y^2$$
$$I_y = \Sigma I_{y0} + \Sigma A \cdot d_x^2$$
$$I_Z = I_x + I_y$$
$$k_x = \sqrt{\frac{I_x}{A}}$$
$$k_y = \sqrt{\frac{I_y}{A}}$$

5.8. Parallel Axis Theorem

$$I_x = I_{xc} + Ad^2$$

$$I_y = I_{yc} + Ad^2$$

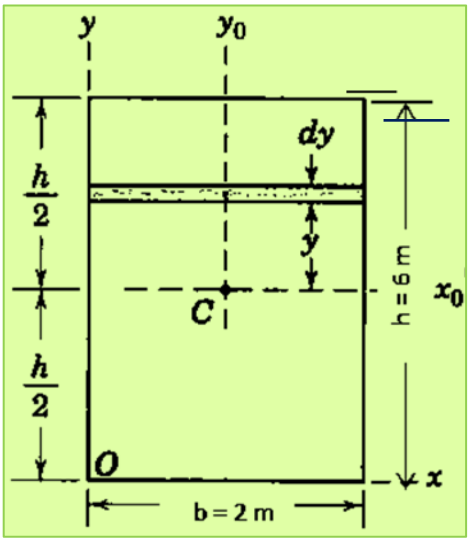
The moment of inertia of an area with respect to any given axis is equal to the moment of inertia with respect to the centroidal axis plus the product of the area and the square of the distance between the 2 axes.

The parallel axis theorem is used to determine the moment of inertia of composite sections.

5.9. Solved Examples

Example 1.

Determine the moments of inertia of the rectangular area about the centoidal (x_0 and y_0) axes, the centroidal polar (z_0) through (C), the x-axes, and the polar axis (z) through (O). If ($h = 6\,m$ and $b = 2\,m$).



Solution:

$$\boldsymbol{A = bh = 2} \times 6 = 12\ m^2$$

By interchange of symbols, the moment of inertia about the centroidal ($x_o - axis$) is:

$$\boldsymbol{I_x = \int y^2 dA}$$

$$\overline{I_x} = \int_{-h/2}^{h/2} y^2 \,.\, b\, dy = b.\left[\frac{y^3}{3}\right]_{-h/2}^{h/2} = b\,.\left[\frac{\left(\frac{h^3}{8}\right)}{3} + \frac{\left(\frac{h^3}{8}\right)}{3}\right] = b\,.\left[\frac{2h^3}{24}\right] = \frac{1}{12} bh^3$$

$$\overline{I_x} = \frac{1}{12} \times 2 \times 6^3 = \frac{432}{6} = 36\, m^4$$

By interchange of symbols, the moment of inertia about the centroidal ($y_o - axis$) is:

$$I_y = \int x^2 dA$$

$$\overline{I_y} = \int_{-b/2}^{b/2} x^2 \,.\, h\, dx = h.\left[\frac{x^3}{3}\right]_{-b/2}^{b/2} = h\,.\left[\frac{\left(\frac{b^3}{8}\right)}{3} + \frac{\left(\frac{b^3}{8}\right)}{3}\right] = h\,.\left[\frac{2b^3}{24}\right] = \frac{1}{12} hb^3$$

$$\overline{I_y} = \frac{1}{12} \times 6 \times 2^3 = \frac{48}{12} = 4\, m^4$$

The centroidal polar of inertia is:

$$\overline{I_z} = \overline{I_x} + \overline{I_y}$$

$$\overline{I_z} = \frac{1}{12} bh^3 + \frac{1}{12} hb^3 = \frac{1}{12} hb(h^2 + b^2) = \frac{1}{12} \times 6 \times 2 \times (2^2 + 6^2) = 40\, m^4$$

By the parallel – axis theorem the moment of inertia about the x-axis is:

$$I_x = \overline{I_x} + Ad_x^2$$

$$I_x = \frac{1}{12} bh^3 + A\left(\frac{h}{2}\right)^2 = \frac{1}{12} \times 2 \times 6^3 + 12 \times \frac{6^2}{4} = 144\ \ m^4$$

Also obtain the polar moment of inertia about (O) by the parallel axis theorem, which gives the following:

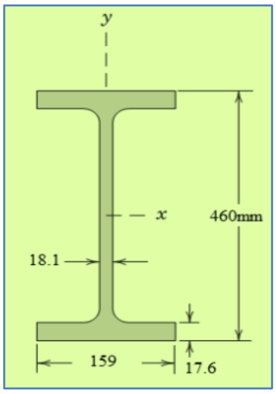
$$I_z = \overline{I_z} + Ad_Z^2$$

$$I_z = \frac{1}{12} A(b^2 + h^2) + A\left[\left(\frac{b}{2}\right)^2 + \left(\frac{h}{2}\right)^2\right] = \frac{1}{3} A(b^2 + h^2)$$

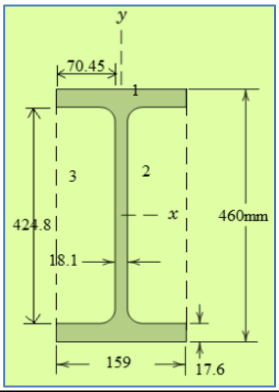
$$= \frac{1}{3} \times 12 \times (2^2 + 6^2) = 272\ \ m^4$$

Example 2.

The cross-sectional area of a wide-flange I-beam has the dimensions shown. Obtain a close approximation to the handbook value of by treating the section as being composed of three rectangles.



Solution:



Part	A (m)	dx (m)	dy (mm)	A , dx^2 (m^4)	A , dy^2 (m^4)	Ix_0 (m^4)	Iy_0 (m^4)
1	73140	0	0	0	0	1289702000	154087695
2	-29927	$\frac{70.45}{2}+\frac{18.1}{2}=44.275$	0	-58665168.63	0	-450042237.91	-12377879.61
3	-29927	$\frac{70.45}{2}+\frac{18.1}{2}=44.275$	0	-58665168.63	0	-450042237.91	-12377879.61
Σ	13286			-117330337.26	0	389617524.18	129.331935.78

$$\boldsymbol{I_x} = \boldsymbol{\Sigma I_{x0}} + \boldsymbol{\Sigma A\,.\,d_y^2} = 389617524.18 + 0 = 389617524.18\,mm^4$$

$$\boldsymbol{I_y} = \boldsymbol{\Sigma I_{y0}} + \boldsymbol{\Sigma A\,.\,d_x^2} = 129.331935.78 - 117330337.26 = 12001598.52\,mm^4$$

$$\boldsymbol{I_Z} = \boldsymbol{I_x} + \boldsymbol{I_y} = 389617524.18 + 12001598.52 = 401619122.7\,mm^4$$

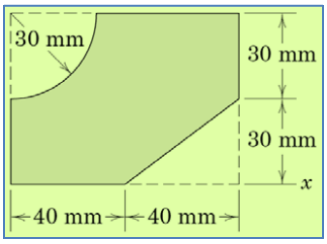
$$\boldsymbol{k_x} = \sqrt{\frac{\boldsymbol{I_x}}{\boldsymbol{A}}} = \sqrt{\frac{389617524.18}{13286}} = 171.25\,mm$$

$$\boldsymbol{k_y} = \sqrt{\frac{\boldsymbol{I_y}}{\boldsymbol{A}}} = \sqrt{\frac{12001598.52}{13286}} = 30.06\,mm$$

$$\boldsymbol{k_z} = \sqrt{\frac{\boldsymbol{I_y}}{\boldsymbol{A}}} = \sqrt{\frac{401619122.7}{13286}} = 173.86\,mm$$

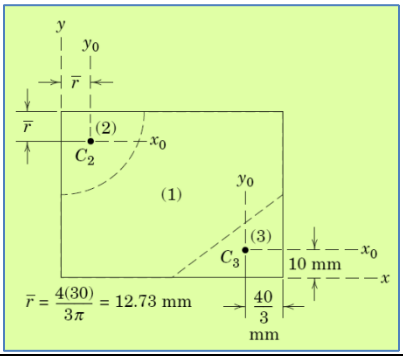
Example 3.

Determine the moments of inertia about the x- and y-axes for the shaded area. Make direct use of the expressions given in Table 1 for the centroidal moments of inertia of the constituent parts.



Solution:

The given area is subdivided into the three subareas shown a rectangular (1), a quarter circular (2), and a triangular (3) area. Two of the subareas are "holes" with negative areas. Centroidal x0 - y0 axes are shown for areas (2) and (3), and the locations of centroids C2 and C3 are from Table 1. The following table will facilitate the calculations.



Part	A (mm)	dx (mm)	dy (mm)	A . dx^2 (mm^3)	A . dy^2 (mm^3)	Ix_0 (mm^4)	Iy_0 (mm^4)
1	4800	40	30	7680000	4320000	1440000	2560000
2	-707.18	12.73	47.27	-114600.57	-1580160.4	-44452.8	-44452.8
3	-600	66.67	10	-2666933.34	-60000	-30000	-53333.33
Total	3492.82			4898466.09	2679839.6	1365547.2	158213.87

$$\boldsymbol{I_x} = \boldsymbol{\Sigma I_{x0}} + \boldsymbol{\Sigma A} \,.\, \boldsymbol{d_y^2} = 1365547.2 + 2679839.6 = 4045386.8\, mm^4$$

$$\boldsymbol{I_y} = \boldsymbol{\Sigma I_{y0}} + \boldsymbol{\Sigma A} \,.\, \boldsymbol{d_x^2} = 158213.87 + 4898466.09 = 7360679.96\, mm^4$$

$$\boldsymbol{I_Z} = \boldsymbol{I_x} + \boldsymbol{I_y} = 4045386.8 + 7360679.96 = 11406066.76\ mm^4$$

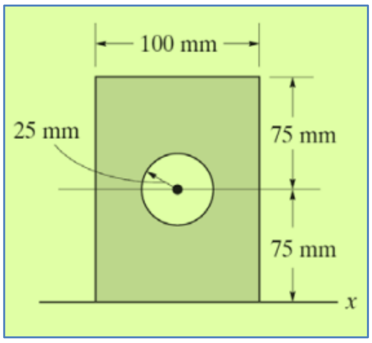
$$\boldsymbol{k_x} = \sqrt{\frac{\boldsymbol{I_x}}{\boldsymbol{A}}} = \sqrt{\frac{4045386.8}{3492.82}} = 34.032\, mm$$

$$\boldsymbol{k_y} = \sqrt{\frac{\boldsymbol{I_y}}{\boldsymbol{A}}} = \sqrt{\frac{7360679.96}{3492.82}} = 45.906\, mm$$

$$\boldsymbol{k_z} = \sqrt{\frac{\boldsymbol{I_y}}{\boldsymbol{A}}} = \sqrt{\frac{11406066.76}{3492.82}} = 57.145\, mm$$

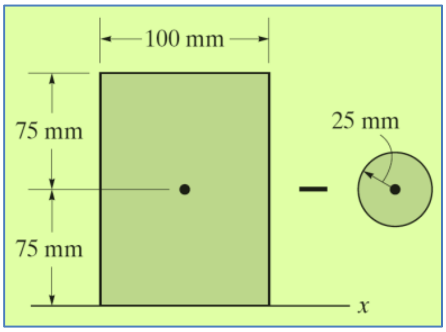
Example 4.

Determine the moment of inertia of the area shown in figure below about the x-axis.



Solution:

Composite Parts. The area can be obtained by subtracting the circle from the rectangle shown in figure below. The centroid of each area is located in the figure. Parallel-Axis Theorem. The moments of inertia about the x-axis are determined using the parallel-axis theorem and the geometric properties formulae for circular and rectangular areas ($Ix = \pi r^4/4$; $Ix = bh^3/12$), found in table 1.



Part	A (mm)	dx (mm)	dy (mm)	A . dx^2 (mm^3)	A . dy^2 (mm^3)	Ix_0 (mm^4)	Iy_0 (mm^4)
1	15000	50	75	37500000	84375000	28125000	12500000
2	-1964.29	50	75	-4910725	-11049131.25	-306919.64	-306919.64
Total	13035.71			32589275	73325868.75	27818080.36	12193080.36

$$I_x = \Sigma I_{x0} + \Sigma A \,.\, d_y^2 = 27818080.36 + 73325868.75 = 101143949.11\, mm^4$$

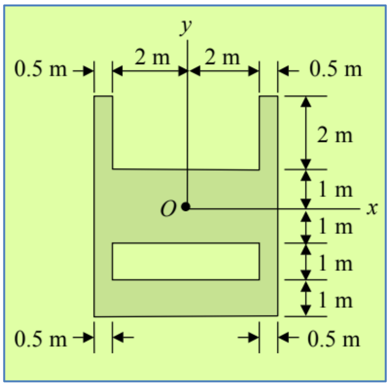
$$I_y = \Sigma I_{y0} + \Sigma A \,.\, d_x^2 = 12193080.36 + 32589275 = 44782355.36\, mm^4$$

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{101143949.11}{13035.71}} = 88.09\, mm$$

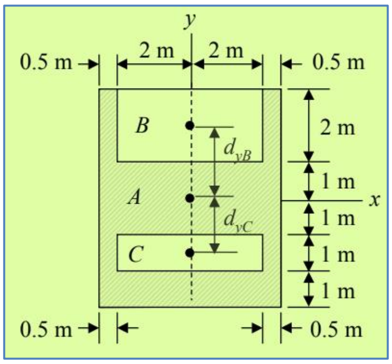
$$k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{44782355.36}{13035.71}} = 58.61 mm$$

Example 5.

Determine the moments of inertia and the radius of gyration of the shaded area with respect to the x and y axes.



<u>Solution:</u>



Part	A (m)	dx (m)	dy (mm)	A , dx^2 (m^4)	A , dy^2 (m^4)	Ix_0 (m^4)	Iy_0 (m^4)
A	30	0	0	0	0	90	62.5
B	-8	0	2	0	-32	-2.67	-10.67
C	-4	0	1.5	0	-9	-0.33	-5.33
Σ	18			0	-41	87	46.5

$$I_x = \Sigma I_{x0} + \Sigma A \,.\, d_y^2 = 87 - 41 = 46mm^4$$

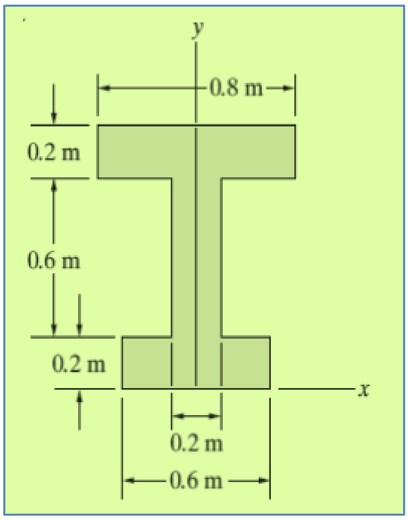
$$I_y = \Sigma I_{y0} + \Sigma A \,.\, d_x^2 = 78.5 - 24 = 54.4\ mm^4$$

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{46}{18}} = 1.599\ mm$$

$$k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{46.5}{18}} = 1.607\ mm$$

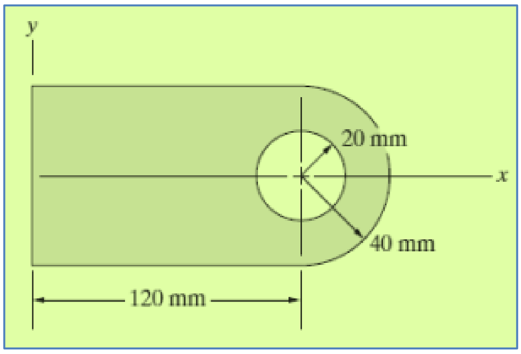
5.10. . Chapter Questions

Q₁: Determine the moment of inertia of the section relative to the x-axis?



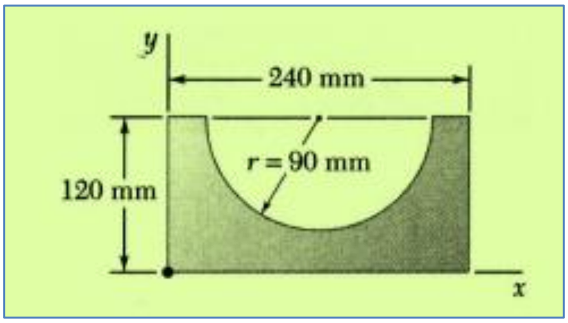
A) I_x = 109.6 (10^9) mm^4
B) I_x = 163.6 (10^9) mm^4
C) I_x = 224.0 (10^9) mm^4
D) I_x = 298.5 (10^9) mm^4

Q_2: Determine the moment of inertia of the section relative to the x-axis?



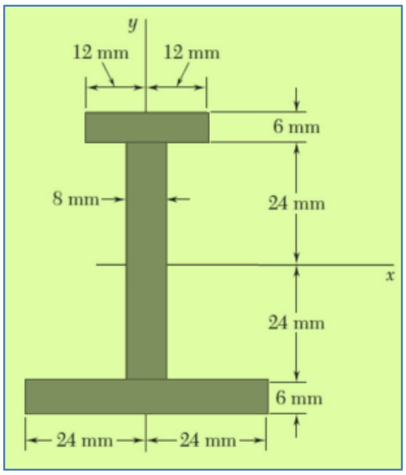
A) I_x = 6.0 (10^6) mm^4
B) I_x = 9.0 (10^6) mm^4
C) I_x = 12.0 (10^6) mm^4
D) I_x = 15.0 (10^6) mm^4

Q_3: Determine the moment of inertia of the shaded area with respect to the x - axis?



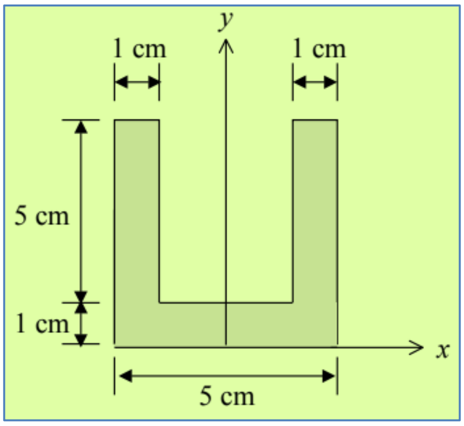
{Answer: $I_x = 92.3 \times 10^6 m^4$ }

Q_4: Determine the moment of inertia of the area shown in the figure?



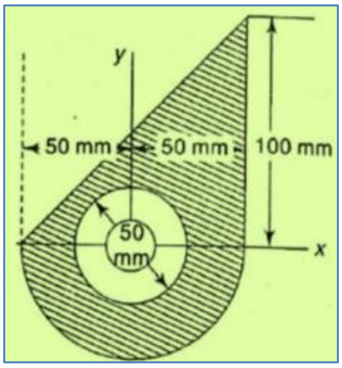
{*Answer:* $I_x = 39 \times 10^4\ mm^4$ }

Q$_5$: Determine the moments of inertia and the radius of gyration of the shaded area with respect to the x and y axes and at the centroidal axes?



{*Answer:* $I_x = 145\ cm^4$, $\breve{I}_x = 51.25\ cm^4$, $\breve{I}_y = 51.25\ cm^4$, $\breve{K}_x = \breve{K}_y = 1.848\ cm$}

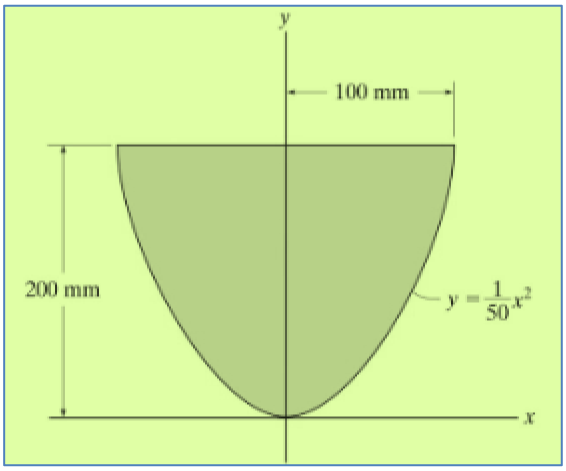
Q$_6$: : Determine the moment of inertia of the area and the radius of gyration shown in the figure?



{*Answer:* $I_x = 10.47 \times 10^6\ mm^4$, $K_x = 1.848\ cm$}

Q$_7$: Determine the moment of inertia of the shaded area with respect to the x – axis and y-axis?

$$y = \frac{1}{50}\,x^2$$



{Answer: $I_x = 45.7 \times 10^6\ mm^4$, $I_y = 53 \times 10^6\ mm^4$}

Chapter 6

Strength of Materials

6. Strength of Materials

6.1. Introduction

Strength of materials, also known as **Mechanics of materials**, is a subject which deals with the behavior of solid subject to stresses and strains.

Stress & Strain

When a force is applied to a structural member, that member will develop both stress and strain as a result of the force.

The applied force will cause the structural member to deform by some length, in proportion to its stiffness.

1. Stress

Stress is the force carried by the member per unit area, and typical units are $[lbf\ /\ in^2\ (psi)]$ for US Customary units and $[N\ /\ m^2\ (Pa)]$ for SI units:

$$\sigma = \frac{F}{A} \qquad (1-1)$$

Where, (F) is the applied force and (A) is the cross-sectional area over which the force acts.

2. Strain

Strain is the ratio of the deformation to the original length of the part:

$$\varepsilon = \frac{L - L_O}{L_O} = \frac{\delta}{L_O} \qquad (1-2)$$

Where (L) is the deformed length, $(\mathrm{L_O})$ is the original unreformed length (ε) is the deformation, and (δ) change in length.

7-2. Types of loading

There are different types of loading which result in different types of stress.

1. Axial Force

Type of stress is called an Axial Stress (general case)

A. Tensile Stress (σ_t): If force is tensile as figure (1-1).

$$\sigma_t = \frac{F}{A} \qquad (1-3)$$

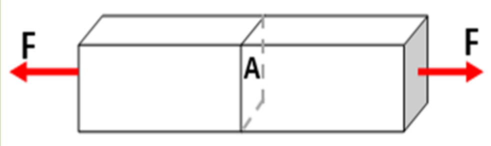


Figure 1-1: Tensile stress

B. Compressive Stress (σ_c): If force is compressive as figure (1-2).

$$\sigma_c = \frac{F}{A} \qquad (1-4)$$

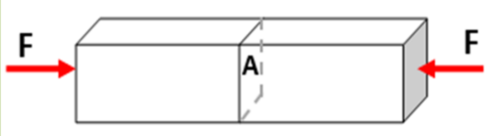


Figure 1-2: Compressive stress

2. Shear stress (τ)

Type of stress is called a Transverse Shear Stress as figure (1-3).

$$\tau = \frac{F}{A} \qquad (1-5)$$

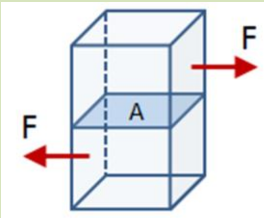


Figure 1-3: Shear stress

3. Bending moment stress (σ_b)

Type of stress is called a Bending Stress as figure (1- 4).

$$\sigma_b = \frac{M \cdot y}{I_c} \qquad (1-6)$$

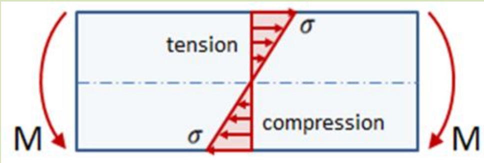


Figure 1-4: Bending stress

Where: (M) is the bending moment, (y) is the distance between the centroid axis and the outer surface, and (I_c) is the centroid moment of inertia of the cross section about the appropriate axis.

4. Torsional stress

Type of stress is called a Torsional Stress as figure (1-5). (**Engineer's theory of Torsion (E.T.T.)).**

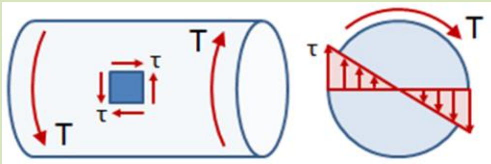


Figure 1-5: Torsional Stress

$$\frac{\tau}{r} = \frac{T}{I} = \frac{G\,.\emptyset}{L} \qquad (1-7)$$

Where: (τ) is the shear stress, (r) is the radius, (T) is the torsion torque, (I) is the polar moment of inertia of the cross section, (G) is modulus of rigidity, $(\emptyset)$ is the torsion angle, and (L) is a length of shaft.

Figure (1-6) shown polar moment of inertia for the following:

$I = \frac{\pi d^4}{32}$ For solid circular section,

$I = \frac{\pi(d_o^4 - d_i^4)}{32}$ For hollow circular section,

$I = \frac{ab^3}{3}$ For solid rectangular section.

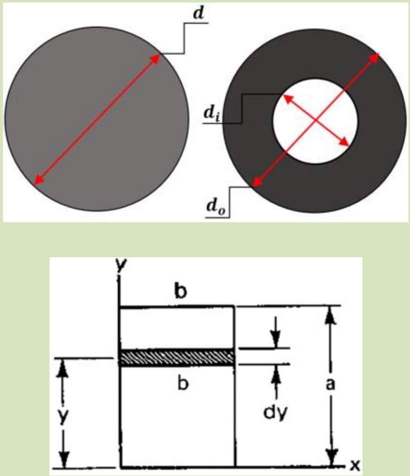


Figure 1-6: Polar moment of inertia

6-3. Hooke's Law

Stress is proportional to strain in the elastic region of the material's stress-strain curve (below the proportionality limit, where the curve is linear), figure (1-6).

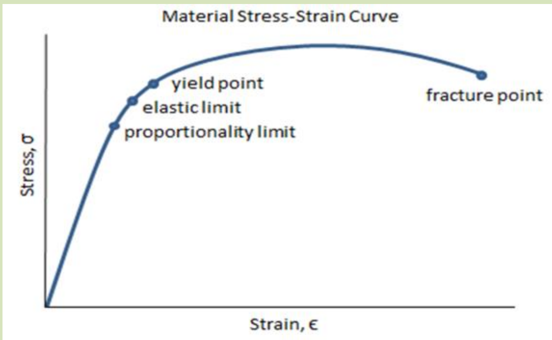


Figure 1-7: Hooke's Law

6-3-1. Engineering and True Stress

6-3-1-1. Engineering Stress (ES)

ES: is equivalent to the applied uniaxial tensile or compressive force at time, a fraction of the specimen's original cross-sectional area, figure (1-8).

1-3-1-2. True Stress (TS)

TS: is equivalent to the applied uniaxial tensile or compressive force at time, divided by the specimen's cross-sectional area at the moment, figure (1-8).

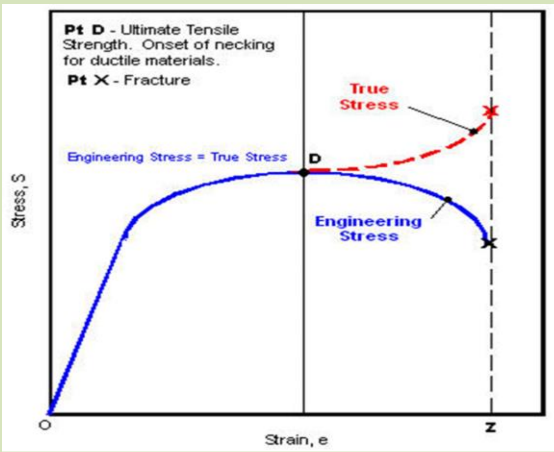


Figure 1-8: Engineering and True Stress

Normal stress and strain are related by:

$$E = \frac{\sigma}{\varepsilon} \qquad (1-8)$$

Where: (E) is the elastic modulus of the material, (σ) is the normal stress, and (ε) is the normal strain.

Shear stress and strain are related by:

$$G = \frac{\tau}{\gamma} \qquad (1-9)$$

Where: (G) is the shear modulus of the material, (τ) is the shear stress, and (γ) is the shear strain. The elastic modulus and the shear modulus are related by:

$$G = \frac{E}{2(1+\mu)} \qquad (1-10)$$

Where: (μ) is Poisson's ratio.

6-4. Poisson's ratio

Poisson's ratio is the proportion of lateral (transverse) contraction strain to longitudinal extension strain in the direction of stretching force, figure (1-9).

The value of Poisson's ratio varies from 0.25 to 0.33. For rubber its value varies from 0.45 to 0.5. Mathematically:

$$Poisson's\ ratio = \frac{Lateral\ strain}{Longitudinal\ strain}$$

$$\mu = \frac{Lateral\ strain}{Longitudinal\ strain}$$

$$\mu = \frac{\varepsilon_{Lateral}}{\varepsilon_{Long.}} \qquad (1-10)$$

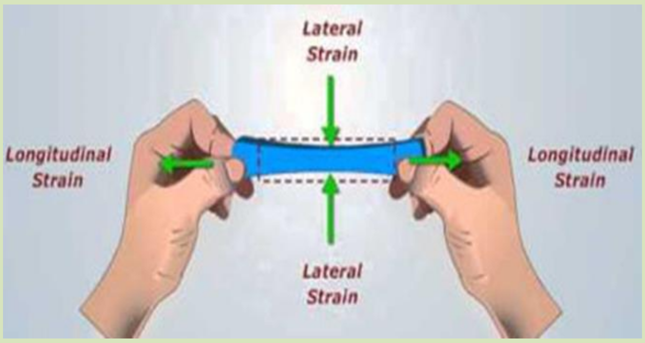


Figure 1-9: Poisson's ratio

6-5. Solve examples

Example 1

A force of (100 KN) is acting on a circular rod, figure (1-10), with diameter (50 mm). The stress in the rod can be calculated as:

Solution:

$\boldsymbol{Given}: F = 100\ KN = 100000\ N, d = 50\ mm, r = \frac{d}{2} = 25\ mm$

$$\boldsymbol{\sigma_t = \frac{F}{A}}$$

$$F = 100 \times 1000 = 100000\ N$$

$$A = \pi\ .\ r^2$$

$$A = \frac{22}{7} \times\ (25)^2 = 1964.286\ mm^2$$

$$\sigma_t = \frac{F}{A} = \frac{100000}{1964.286} = 50.909\ \frac{N}{mm^2} (MPa)$$

Figure 1-10: Circular rod

Example 2

A metal shaft diameter (12 mm), and long (1.5 m), figure (1-11). A tensile force of (1000 N) is applied to it and it stretches (0.11 mm). Assume the material is elastic. Determine the stress and strain in the shaft?

Figure 1-11: A metal shaft

Solution:

$\boldsymbol{Given}: d = 12\ mm\ , r = 6\ mm, L = 1.5\ m = 1500\ mm, F = 1000\ N, \delta = 0.11\ mm$

$$A = \pi\ .\ r^2 = \frac{22}{7} \times (6)^2 = 113.143\ mm^2$$

$$\sigma_t = \frac{F}{A} = \frac{1000}{113.143} = 8.818\ MPa$$

$$\varepsilon = \frac{\delta}{L_O} = \frac{0.11}{1500} = 0.000073 = 73\ \ \mu\varepsilon$$

Example 3

A steel tensile test specimen, figure (1-12) has an across sectional area of (120 mm^2), and gauge length (50 mm), the gradient of elastic section is (433 KN/mm). Determine the modulus of elasticity?

Figure 1-12: A steel tensile test specimen

Solution:

$$\boldsymbol{Given}: A = 120\, mm^2\,, \qquad L = 50\, mm,$$
$$Gradient\ ratio\ (\frac{F}{\delta}) = 433\frac{KN}{mm} = 433000\ N/mm$$

$$E = \frac{\sigma}{\varepsilon} = \frac{F}{\delta}\,.\frac{L}{A} = 433000 \times \frac{50}{120} = 180416.667\, MPa \approx 180.417\ GPa$$

Example 4

A long of the steel column is (4 m), and diameter (50 cm), figure (1-13). It carries a load of (100 MN). If modulus of elasticity is (210 GPa), calculate the compressive stress and strain and how much the column is compressed?

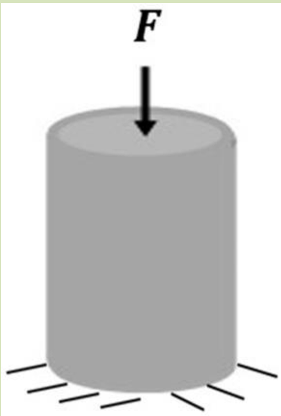


Figure 1-13: A steel column

Solution:

$$\boldsymbol{Given}: L = 4\, m = 4000\, mm,$$
$$d = 50\, cm = 500\, cm, r = 250\, mm, F = 100\, MN = 100000000\, N, E = 210\, GPa = 210000m$$

$$A = \frac{22}{7} \times \ (250)^2 = 196428.571\, mm^2$$

$$\sigma_c = \frac{F}{A} = \frac{100000000}{196428.571} = 509.091\, MPa$$

$$E = \frac{\sigma}{\varepsilon} \qquad \Rightarrow \qquad \varepsilon = \frac{\sigma}{E} = \frac{509.091}{210000} \approx 0.00242 = 24.2\ \mu\varepsilon$$

$$\varepsilon = \frac{\delta}{L} \qquad \Rightarrow \qquad \delta = \varepsilon\,.\,L = 0.00242 \times 4000 = 9.68\, mm$$

Example 5

Calculate the force needed to a plate of metal (5 mm) thick and (0.8 m) wide given that the ultimate shear stress (50 MPa), as shown in the figure (1-14)?

Figure 1-14: A plate of metal

Solution:

The area to be cut is a rectangle

$$t = 5\,mm\,;\, w = 0.8\,m = 0.8 \times 1000 = 800\,mm\,;$$

$$\tau = 50\,N / mm^2$$

$$A = w.t = 5 \times 800 = 4000\ mm^2$$

$$\because\ \tau = \frac{F}{A} \quad \Rightarrow \quad F = \tau.A = 50 \times 4000 = 200000N = 2$$

Example 6

Calculate the force needed to shear a Screw (12 mm) diameter given that the ultimate shear stress is (90 MPa), as shown in the figure (1-15)?

Figure 1-15: A plate of metal

Solution:

The area to be is the circular area:

$$A = \frac{\pi\, d^2}{4} = \frac{3.14 \times (12)^2}{4} = 113.04\ mm^2$$

$$\tau = \frac{F}{A}$$

$$F = \tau.A = 90 \times 113.04 = 10173.6\ N \approx 10.17\ KN$$

Example 7

A pin is used to attach a clevis to a rope, figure (1-16). The force in the rope will be a maximum of (60 KN). The maximum permitted shear stress in a pin is (40 MPa). Calculate the diameter of suitable pin?

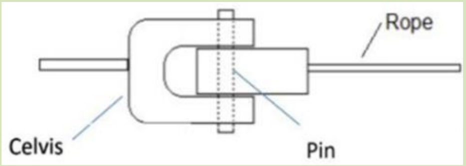


Figure 1-16: A clevis to a rope

Solution

The pin is in double shear so the shear stress is:

$$A = \frac{F}{2\tau} = \frac{60000}{2 \times 40} = 750\ mm^2$$

Also:

$$d = \sqrt{\frac{4A}{\pi}} = \sqrt{\frac{4 \times 750}{3.143}} = 30.9\ mm$$

Example 8

A simply supported beam is subject a point load of (200 N) at the mid - spam of the beam as shown in the figure (1-17). The beam has a circular (50 mm) diameter. Calculate the maximum stress due to bending?

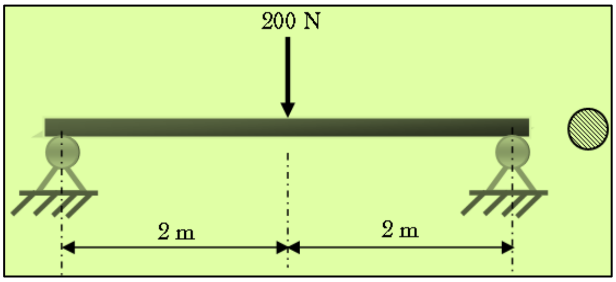
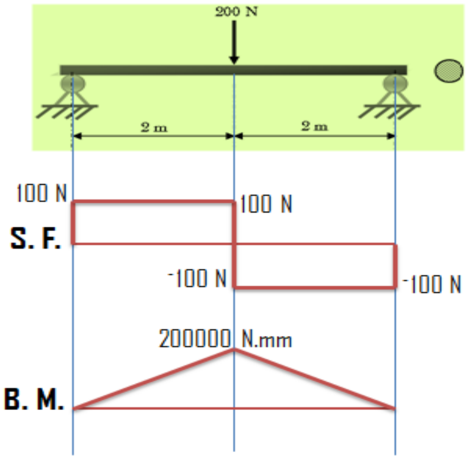


Figure 1-17: A simply supported beam

Solution:

Given:

$$F = 200\ N, d = 50\ mm\ , L_1 = 2\ m = 2000\ mm\ , L_2 = 2\ m = 2000\ mm.$$



$$\sigma_{max.} = \frac{M\ .C}{I}$$

$$M = 100 \times 2000 = 200000\ N.mm$$

$$I = \frac{\pi\ d^4}{64} = \frac{3.143 \times (50)^4}{64} = 306933.59\ mm^4$$

$$C = \frac{d}{2} = \frac{50}{2} = 25\ mm$$

$$\boldsymbol{\sigma_{max.}} = \frac{\boldsymbol{M\ .C}}{\boldsymbol{I}} = \frac{200000 \times 25}{306933.59} = 16.29\ \text{MPa}$$

Example 9

A diameter solid steel shaft (ABCDE), figure (1-18) is (50 mm) see in figure. If have torques (T_1 = 200 N. m, T_2 = 500 N. m and T_3 = 300 N. m), distance between gears (B & C) is (L1 = 200 mm) and distance between gears (C & D) is (L_2 = 300 mm), modulus of rigid is (G = 90 GPa). Determine the maximum shear stress ($\tau_{max.}$) in each part and twisting angle (ϕ_{BD})?

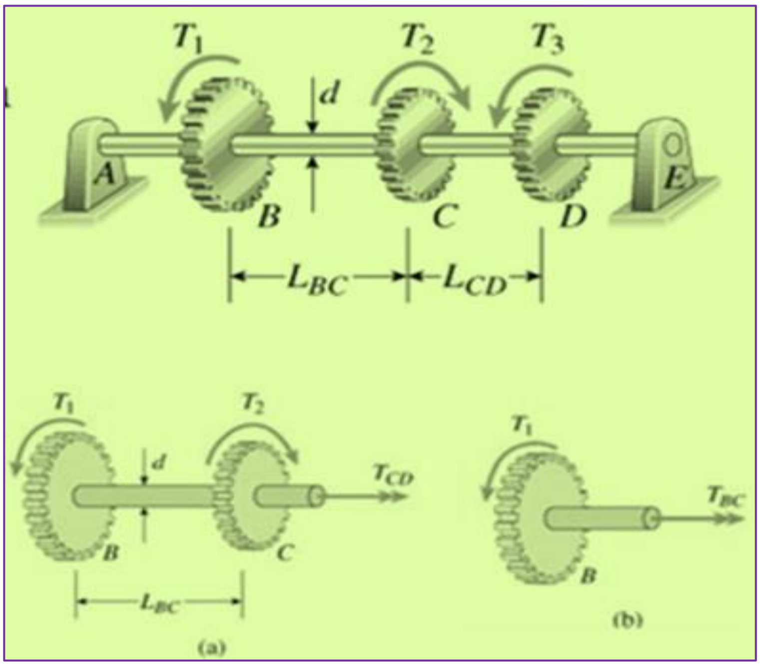


Figure 1-18: A simply supported beam

Solution

Given: *{d= 50 mm, T_1 = 200 N. m, T_2 = 500 N .m, T_3 = 300 N. m, L_1 = 300 mm, L_2 = 200 mm, G = 90 GPa}.*

$$I_P = \frac{\pi . d^4}{32} = \frac{3.14 \times 50^4}{32} = 613281.25 mm^4 ; \quad r = 25 mm$$

$$T_{BC} = -T_1 = -200 \ N.m$$

$$T_{CD} = T_2 - T_1 = 500 - 200 = 300 N.m$$

$$\because \frac{\tau}{r} = \frac{T}{I} = \frac{G\varphi}{L} \qquad \Rightarrow \qquad \therefore \tau = \frac{T.r}{I}$$

$$\therefore \tau_{BC} = \frac{T_{BC}.r}{I} = \frac{200.10^3 \times 25}{613281.25} = 8.15 \ MPa$$

$$\therefore \tau_{CD} = \frac{T_{CD}.r}{I} = \frac{300.10^3 \times 25}{61328125} = 12.23\ MPa$$

$$\phi_{BD} = \phi_{BC} - \phi_{CD}$$

$$\because \frac{\tau}{r} = \frac{T}{I} = \frac{G\varphi}{L} \qquad \Rightarrow \qquad \therefore \varphi = \frac{T.\,L}{I.G}$$

$$\varphi_{BC} = \frac{T_{BC}.\,L_1}{I.G} = \frac{200.10^3 \times 300}{61328125 \times 90.10^3} \approx 0.00109 \approx 0.0624^\circ$$

$$\varphi_{CD} = \frac{T_{CD}.\,L_2}{I.G} = \frac{300.10^3 \times 200}{61328125 \times 90.10^3} \approx 0.00109 \approx 0.0624^\circ$$

Example 10

A steel wire having cross sectional area (2 mm^2), figure (1-19). Is stretched by (200 N). Find the lateral strain produced in the wire. If modulus elasticity for steel is (210 GPA) and Poisson's ratio is (0.233)?

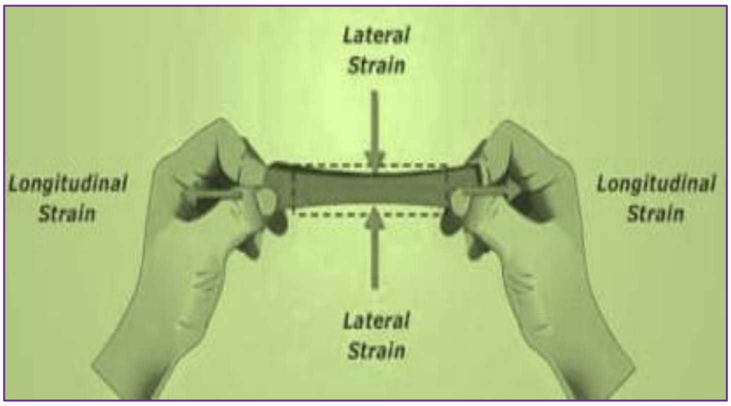


Figure 1-19: A steel wire

Solution

Given: *{A= 2 mm^2, F = 200 N, μ = 0.233, G = 210 GPa}.*

$$E = \frac{\sigma}{\varepsilon_{longitudinal}} = \frac{F}{A.\varepsilon_L} \quad \Rightarrow \ \varepsilon_L = \frac{F}{A.E} = \frac{200}{2.10^{-6} \times 210.10^9} = 0.00048 \approx 4.8 \times 10^4$$

$$\because \ \mu = \frac{\varepsilon_{lateral}}{\varepsilon_{longitudinal}} \qquad \Rightarrow \ \therefore \varepsilon_{Lateral} = \mu.\varepsilon_L = 0.233 \times 0.00048 = 0.000112 \approx 1.12 \times 10^4$$

6.6. Shear Force and Bending Moment Diagram

The maximum absolute value of the shear force and the bending moment of the beams with regard to the relative load can be determined using the shear force and bending moment diagram in beams.

Before we can design the Shear force and bending moment diagram, we must first understand the various types of beams and loads, as well as the reaction forces acting on them.

According to the right or left of the section, the bending moment is the algebraic sum of all the moment of forces. It is the reaction that is induced in a structural element as a result of an external force or moment.

The moment caused by external forces is balanced in the equilibrium position by the couple induced by the internal load; this internal couple is known as a bending moment.

Type of Beams

a. Cantilever Beam

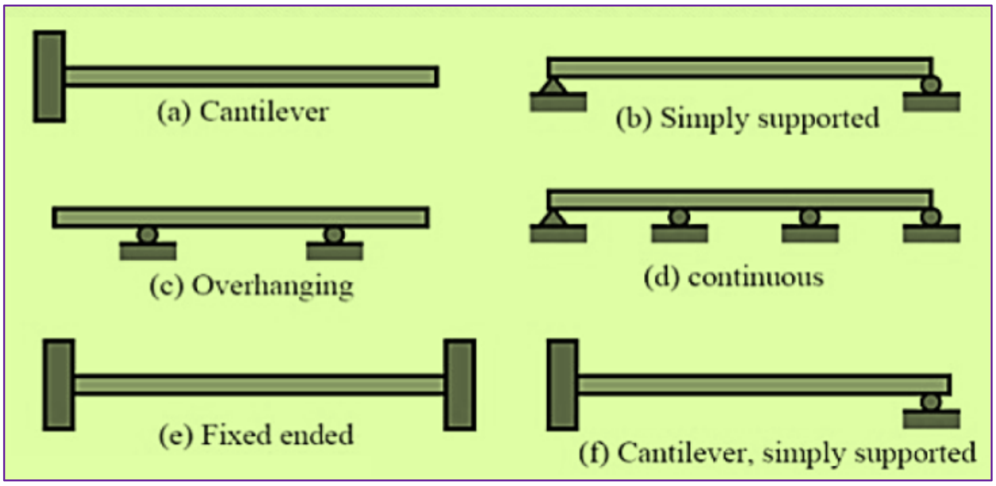
b. Simply Supported Beam

c. Overhanging Beam

d. Continuous Beam

e. Fixed ended Beam

f. Cantilever, Simply Supported Beam



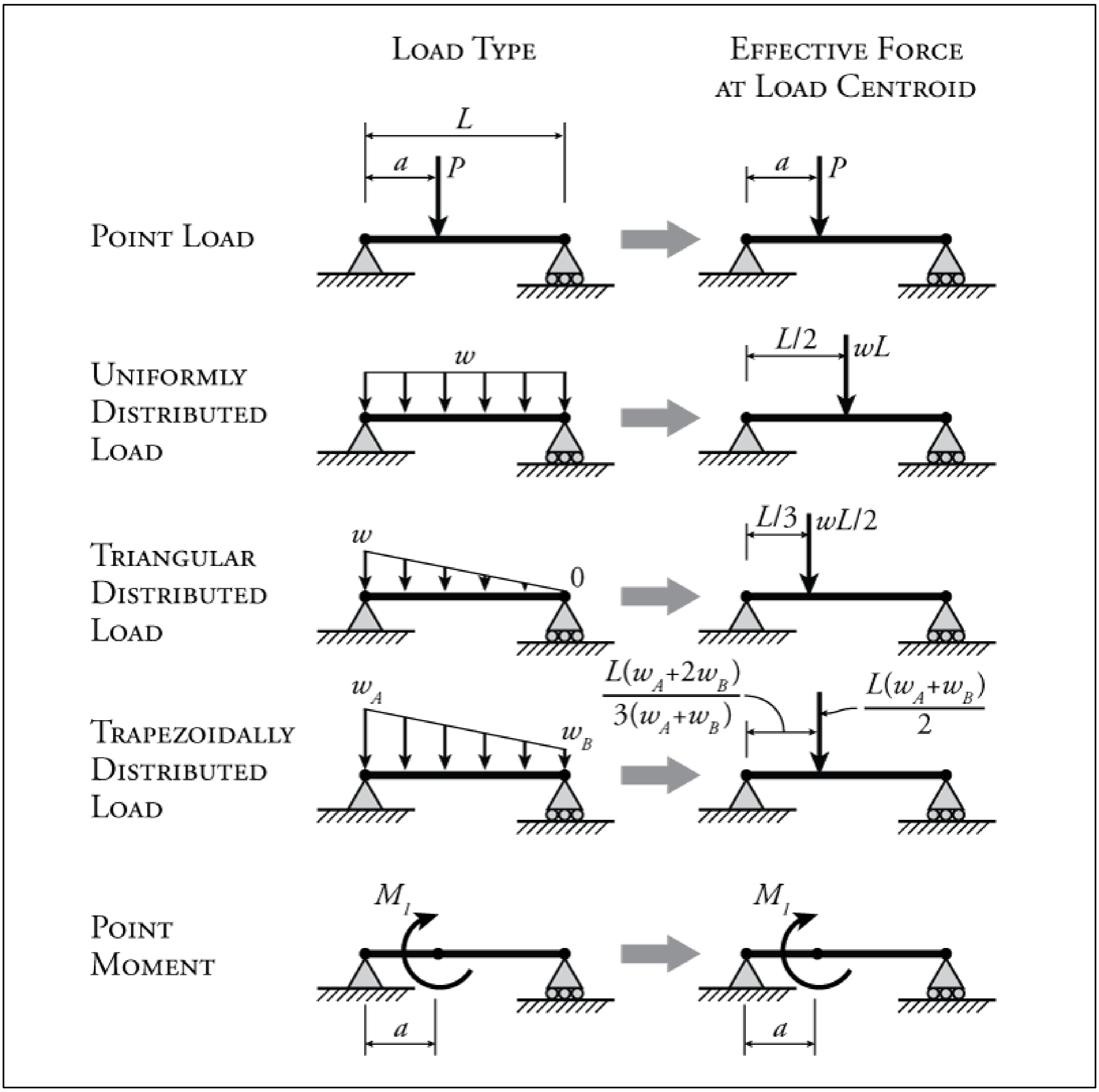
Type of Loads

The applied weight is normally vertical, whereas the beam is usually horizontal.

Concentrated or Point Load: Act at a point.

Uniformly Distributed Load: The load is evenly distributed along the length of the Beam.

Uniformly Varying Load: Load distribution along the length of the beam, and rate of varying loading from point to point.

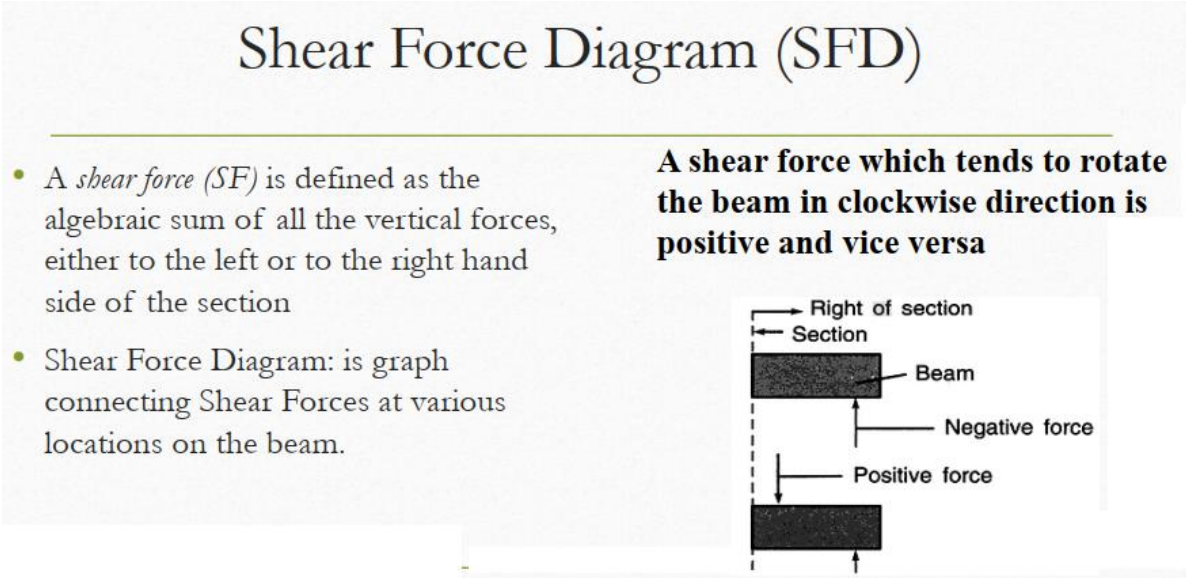


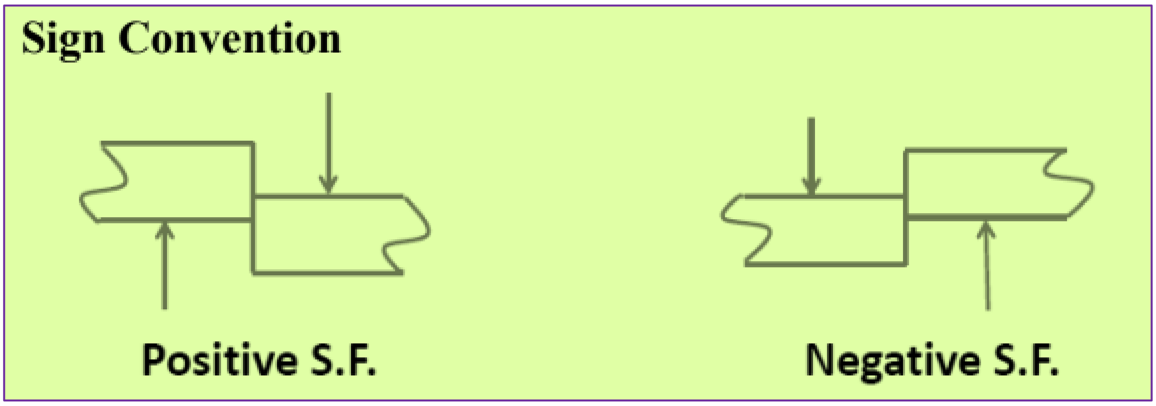
Sign Convention of Shear Force:

Shear force is an imbalanced vertical force that causes one end of the beam to move forward or downward in relation to the other.

When the left-hand portion of a section tends to slide upward and the right-hand half tends to slide downward, the shear force is deemed positive.

When the left-hand portion of a section tends to slide downward, or the right-hand portion tends to slide upward, the shear force at that section is negative.



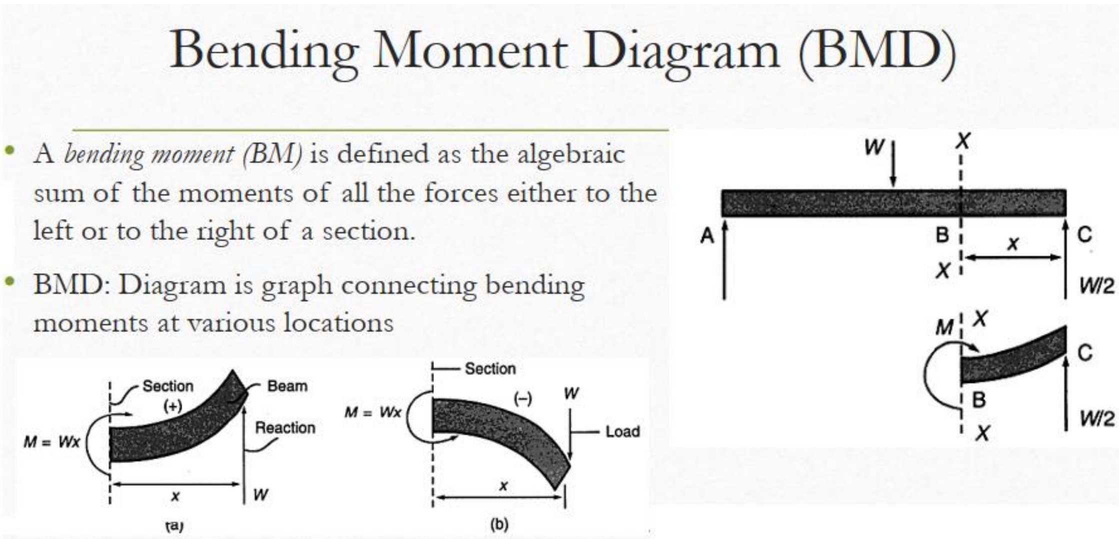


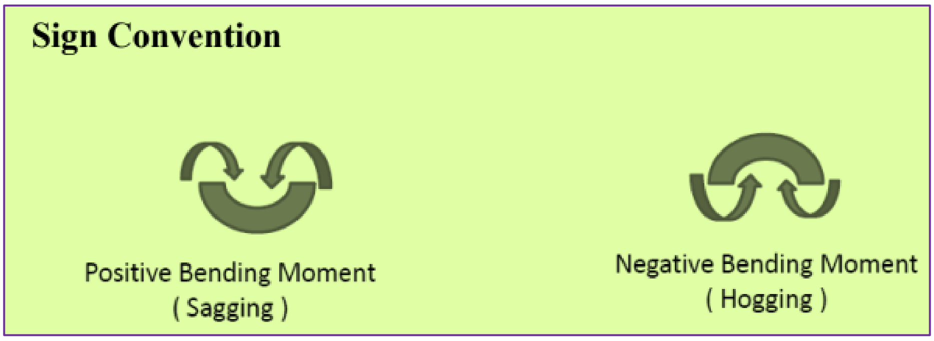
Sign Convention for Bending Moment:

When the bending moment at a section tends to bend the beam at a point to curvature with a concavity at the top, or when the moments are operating clockwise to the left or anti-clockwise to the right, we consider it positive.

On the other hand, the **bending moment** at a section is deemed negative when it tends to bend the beam at a point to curvature with convexity at the top or when moments are taken in an anti-clockwise or clockwise manner.

Positive bending moments are sometimes referred to as sagging moments, whereas negative bending moments are referred to as hogging moments.

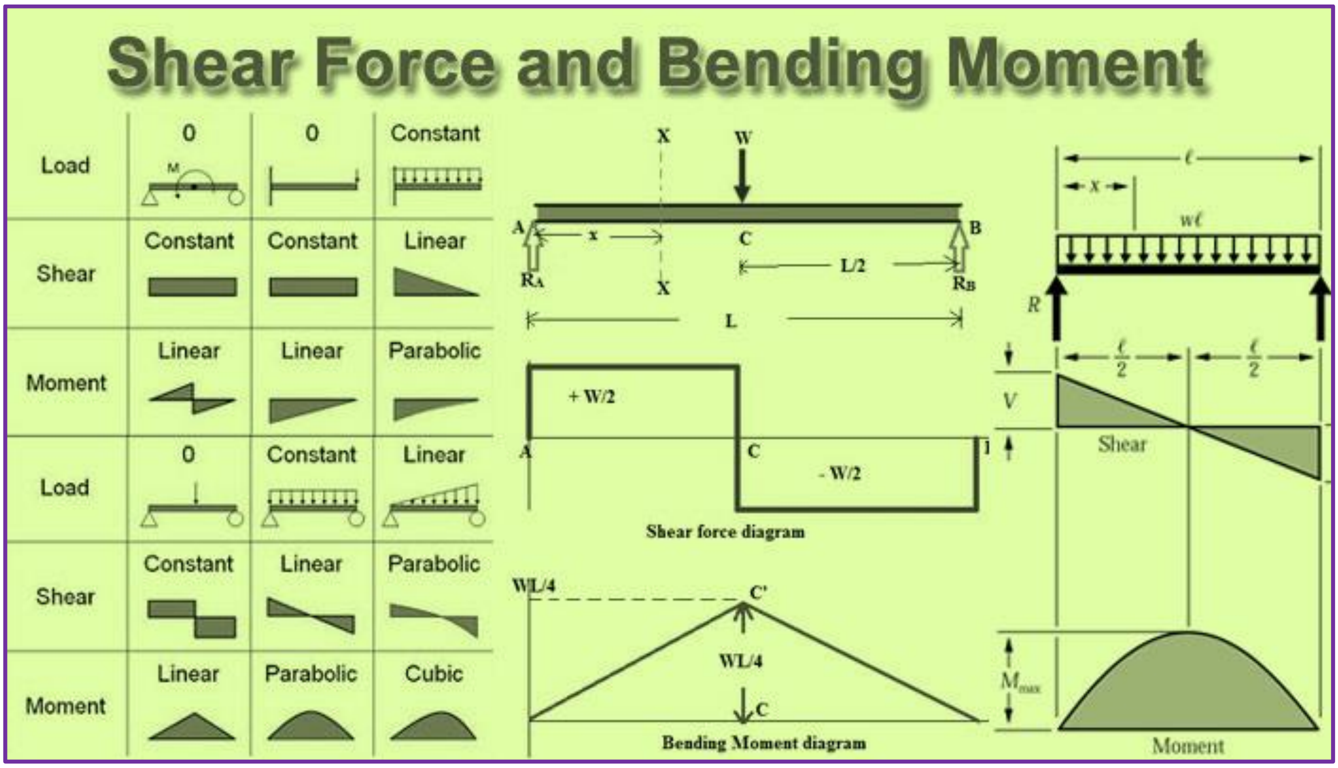




Shear Force and Bending Moment Diagram Drawing Instructions

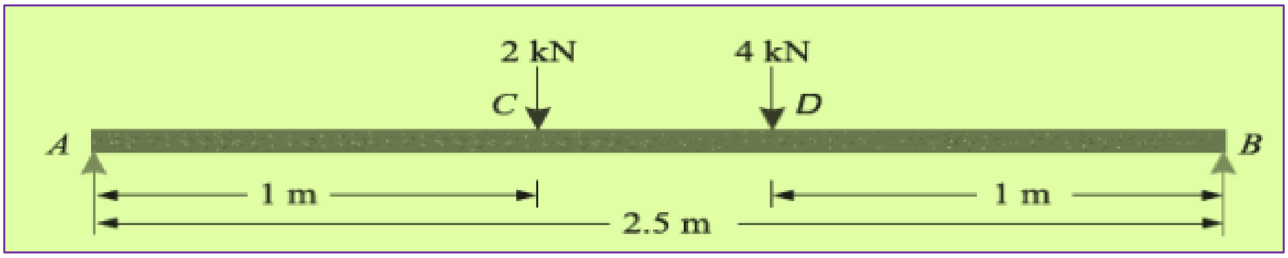
- The ordinates in **SFD and BMD diagrams are shear force or bending moment**, and the abscissa is the length of the beam.
- Take a look at the left or right side of the section.
- On one of the portions, add the forces (including reactions) normal to the beam.
- The force acting downhill is positive, whereas the force acting upwards is negative if the right portion of the section is chosen.

- The force acting downhill is negative, whereas the force acting upwards is positive if the Left component of the section is chosen.
- Shear force and Bending moment positive values are plotted above the baseline, while negative values are plotted below the baseline.
- The sheer force diagram will suddenly increase or decrease. I.e., at a segment where there is a vertical point load, by a vertical straight line.
- Between any two vertical weights, the shear force will be constant. As a result, the horizontal shear force between the two vertical loads will exist.
- At the two ends of a simply supported beam and at the free end of a cantilever, the bending moment will be zero.



Example 1.

Draw shear force and bending moment of a simply supported beam (AB) shown in figure, of span (2.5 m) is carrying two point loads as.



Solution:

1. Reactions

$$\curvearrowright_{+}\ \mathbf{\Sigma M_A = 0}$$

$$2 \times 1 + 4 \times 1.5 - R_B \times 2.5 = 0$$

$$R_B = \frac{2 \times 1 + 4 \times 1.5}{2.5} = \frac{8}{2.5} = 3.2\ KN$$

$$\curvearrowright_{+}\ \mathbf{\Sigma M_B = 0}$$

$$4 \times 1 + 2 \times 1.5 - R_A \times 2.5 = 0$$

$$R_A = \frac{4 \times 1 + 2 \times 1.5}{2.5} = \frac{7}{2.5} = 2.8\ KN$$

Or

$$\mathbf{\Sigma F_y = 0}$$

$$R_A - 2 - 4 + R_B = 0$$

$$R_A = 2 + 4 - 3.2 = 2.8\ KN$$

2. Shear force diagram:

$$\Sigma F_A = 2.8 \text{ KN}$$

$$\Sigma F_C = 2.8 - 2 = 0.8 \text{ KN}$$

$$\Sigma F_D = 2.8 - 2 - 4 = -3.2 \text{ KN}$$

$$\Sigma F_B = 2.8 - 2 - 4 + 3.2 = 0 \text{ KN}$$

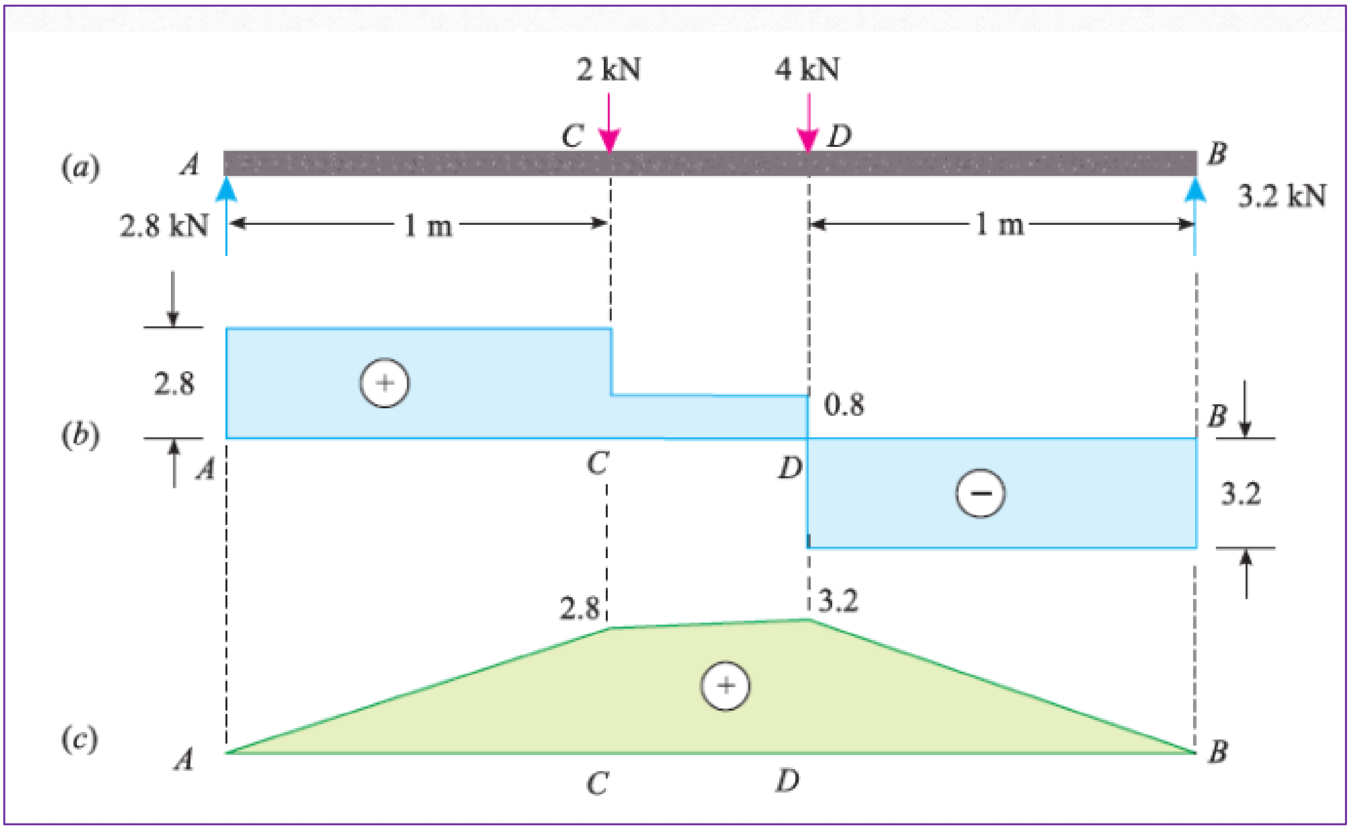
3. Bending moment:

$$\Sigma M_A = 0 \text{ KN.m}$$

$$\Sigma M_C = 2.8 \times 1 = 2.8 \text{ KN.m}$$

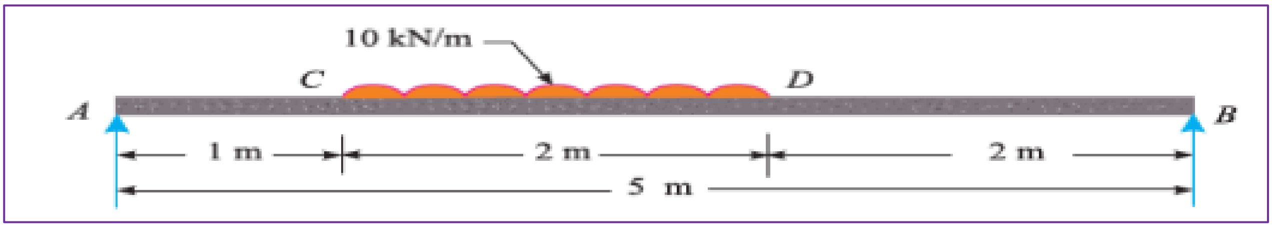
$$\Sigma M_D = 2.8 \times 1.5 - 2 \times 0.5 = 3.2 \text{ KN.m}$$

$$\Sigma M_B = 2.8 \times 2.5 - 2 \times 1.5 - 4 \times 1 = 0 \text{ KN.m}$$



Example 2.

Draw shear force and bending moment of a simply supported beam (AB) shown in figure, of span (5 m) is carrying two point loads as.



Solution:

1. Reactions

$$\curvearrowright_{+}\ \boldsymbol{\Sigma M_A = 0}$$

$$10 \times 2 \times 2 - R_B \times 5 = 0$$

$$R_B = \frac{10 \times 2 \times 2}{5} = \frac{40}{5} = 8\ KN$$

$$\curvearrowright_{+}\ \boldsymbol{\Sigma M_B = 0}$$

$$10 \times 2 \times 3 - R_A \times 5 = 0$$

$$R_A = \frac{10 \times 2 \times 3}{5} = 12\ KN$$

Or

$$\boldsymbol{\Sigma F_y = 0}$$

$$R_A - 10 \times 2 + R_B = 0$$

$$R_A = 20 - 8 = 12\ KN$$

2. Shear force diagram:

$$\text{S.F. at point A} = 12 \text{ KN}$$

$$\text{S.F. at point C} = 12 \text{ KN}$$

$$\text{S.F. at point D} = 12 - 10 \times 2 = -8 \text{ KN}$$

$$\text{S.F. at point A} = 12 - 10 \times 2 + 8 = 0\ KN$$

3. Bending moment:

$$B.M.\ at\ point\ A = 0 \text{ KN.m}$$

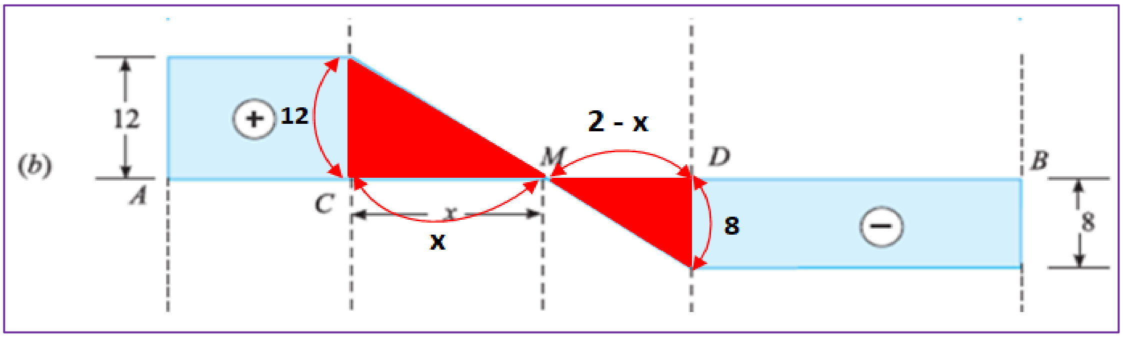
$$B.M.\ at\ point\ C = 12 \times 1 = 12 \text{ KN.m}$$

$$B.M.\ at\ point\ D = 12 \times 3 - 2 \times 10 \times 1 = 16 \text{ KN.m}$$

$$B.M.\ at\ point\ B = 12 \times 5 - 10 \times 2 \times 3 = 0 \text{ KN.m}$$

We follow the following steps to find bending moment at point (M):

From the figure of shear force:



$$\text{S.F. at point M} = 0 \text{ KN}$$

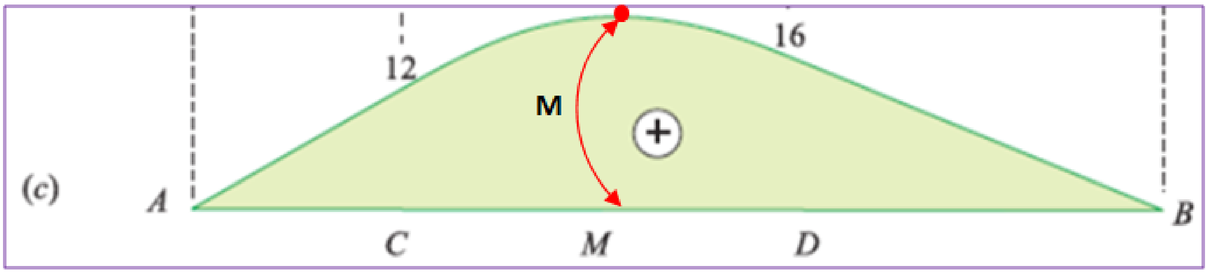
$$\frac{\text{x}}{12} = \frac{2 - x}{8}$$

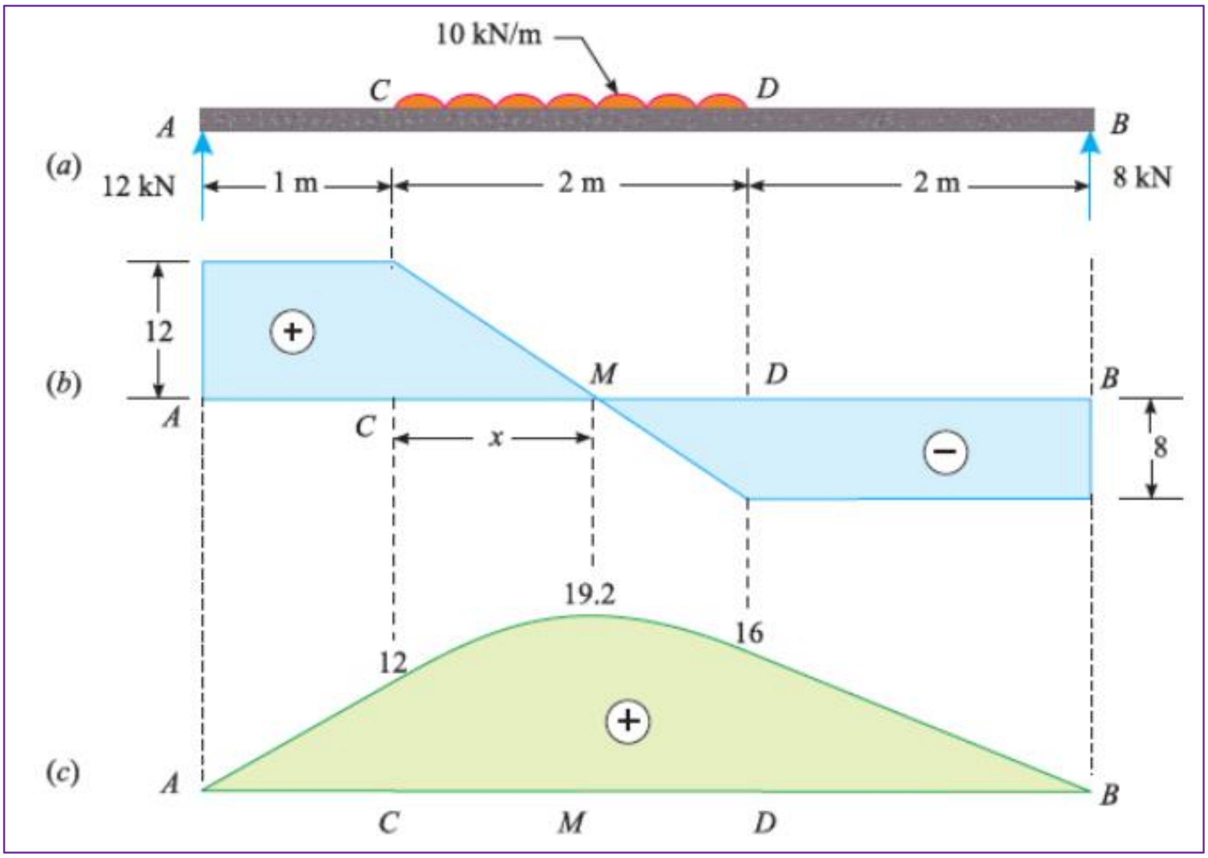
$$8\text{ x} = 24 - 12\,x$$

$$\text{x} = \frac{24}{20} = 1.2\ m$$

The bending moment: $\mathrm{B.M.}$ at point M

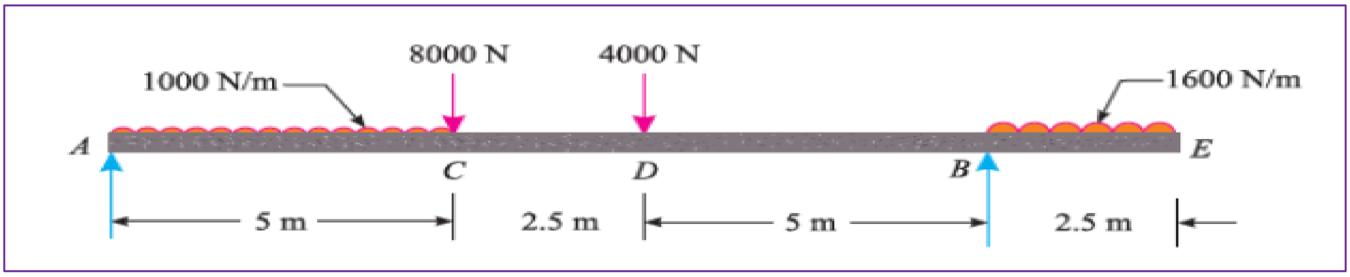
$$B.M.\ at\ point\ B = 12 \times (1 + 1.2) - 10 \times 1.2 \times \frac{1.2}{2} = 19.2\ \mathrm{KN.m}$$





Example 3.

Draw shear force and bending moment of a simply supported beam (AB) shown in figure, of span (15 m) is carrying two point loads as.



Solution:

1. Reactions

$$\curvearrowright_{+}\ \boldsymbol{\Sigma M_A = 0}$$

$$1000 \times 5 \times 2.5 + 8000 \times 5 + 400 \times 7.5 - R_B \times 12.5 + 1600 \times 2.5 \times 13.75 = 0$$

$$R_B = \frac{1000 \times 5 \times 2.5 + 8000 \times 5 + 4000 \times 7.5 + 1600 \times 2.5 \times 13.75}{12.5}$$

$$= \frac{137500}{12.5} = \boldsymbol{11000\ N}$$

$$\boldsymbol{\Sigma F_y = 0}$$

$$R_A - 1000 \times 5 - 8000 - 4000 + R_B - 1600 \times 2.5 = 0$$

$$R_A = 1000 \times 5 + 8000 + 4000 - 11000 + 1600 \times 2.5 = \boldsymbol{10000N}$$

2. Shear force diagram:

$$\text{S.F. at point A} = 10000\ \text{N}$$

$$\text{S.F. at point C} = 10000 - 1000 \times 5 - 8000 = -3000\ \text{N}$$

$$\text{S.F. at point D} = 10000 - 1000 \times 5 - 8000 - 4000 = -7000\ \text{N}$$

$$\text{S.F. at point B} = 10000 - 1000 \times 5 - 8000 - 4000 + 11000 = 4000\ N$$

$$\text{S.F. at point B} = 10000 - 1000 \times 5 - 8000 - 4000 + 11000 - 1600 \times 2.5 = 0\ N$$

3. Bending moment:

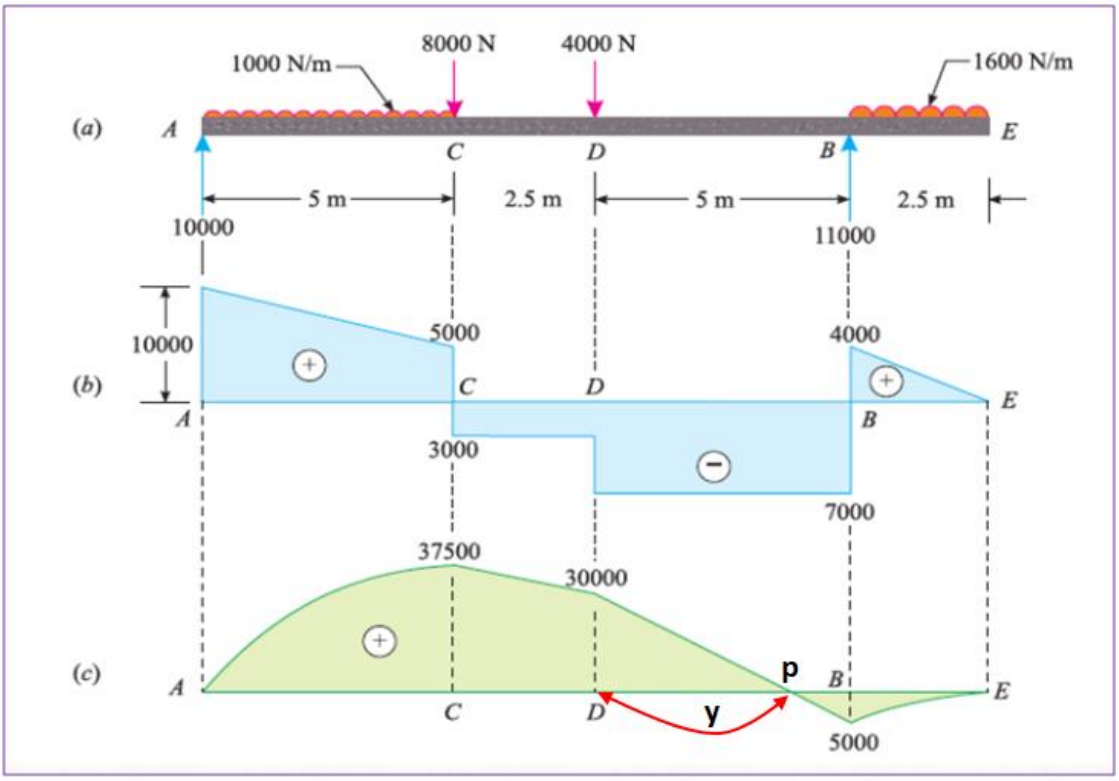
$$B.M.\ at\ point\ A = 0\ \text{KN.m}$$

$$B.M.\ at\ point\ C = 10000 \times 5 - 1000 \times 5 \times 2.5 = 37500\ \text{N.m}$$

$$B.M.\ at\ point\ D = 10000 \times 7.5 - 1000 \times 5 \times 5 - 8000 \times 2.5 = 30000\ \text{N.m}$$

$$B.M.\ at\ point\ B = 10000 \times 12.5 - 1000 \times 5 \times 10 - 8000 \times 7.5 - 4000 \times 5 = -5000\ \text{N.m}$$

$$B.M.\ at\ point\ E = 0\ \text{N.m}$$



To locate the point (p) of contra flexure:

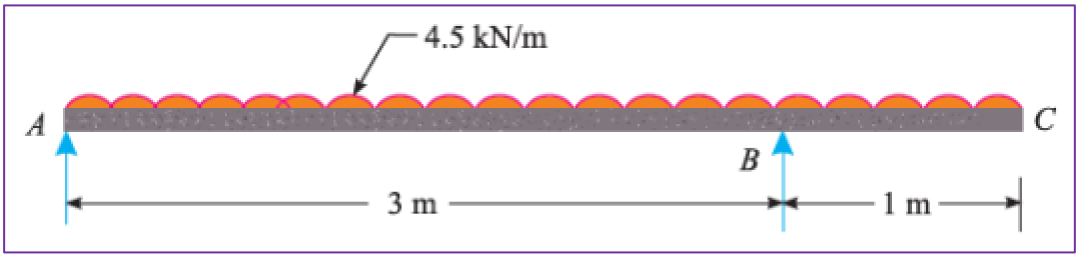
$$\frac{y}{30000} = \frac{5 - y}{5000}$$

$$5000\ y = 150000 - 30000\ y$$

$$y = \frac{150000}{35000} = 4.29\ m$$

Example 4.

Draw shear force and bending moment of a simply supported beam (AB) shown in figure, of span (4 m) is carrying two point loads as.



Solution:

1. Reactions

$$\curvearrowright_{+}\ \boldsymbol{\Sigma M_A = 0}$$

$$4.5 \times 4 \times 2 - R_B \times 3 = 0$$

$$R_B = \frac{4.5 \times 4 \times 2}{3} = \frac{36}{3} = 12\ KN$$

$$\curvearrowright_{+}\ \boldsymbol{\Sigma M_B = 0}$$

$$3.5 \times 3 \times 1.5 - R_A \times 3 + 4.5 \times 1 \times 0.5 = 0$$

$$R_A = \frac{3.5 \times 3 \times 1.5 + 4.5 \times 1 \times 0.5}{3} = 6\ KN$$

$$\boldsymbol{\Sigma F_y = 0}$$

$$R_A - 4.5 \times 4 + R_B = 0$$

$$R_A = 18 - 12 = 6\ KN$$

2. Shear force diagram:

$$\text{S.F. at point A} = 6\ \text{KN}$$

$$\text{S.F. after point B} = 6 - 4.5 \times 3 = -7.5\ \text{KN}$$

$$\text{S.F. at point B} = 6 - 4.5 \times 3 + 12 = 4.5 \text{ KN}$$

$$\text{S.F. at point C} = 6 \times 4 - 4.5 \times 4 + 12 = 0 \text{ KN}$$

3. Bending moment:

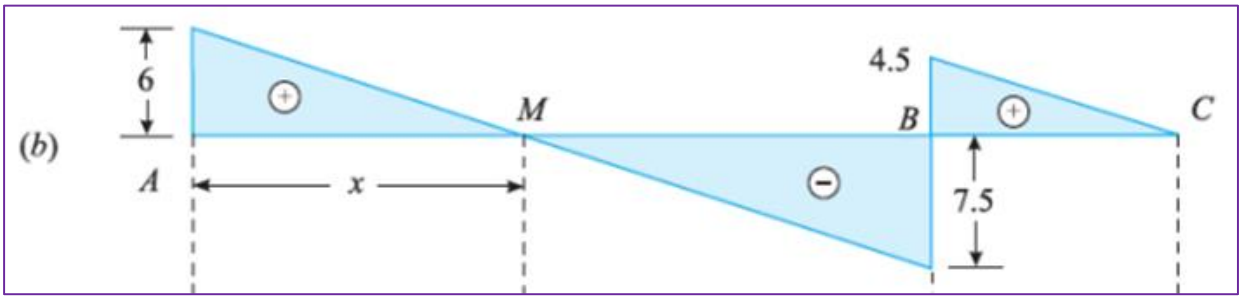
$$B.M.\ at\ point\ A = 0 \text{ KN.m}$$

$$B.M.\ at\ point\ B = 6 \times 3 - 4.5 \times 3 \times 1.5 = -2.25 \text{ KN . m}$$

$$B.M.\ at\ point\ C = 0 \text{ KN . m}$$

We follow the following steps to find bending moment at point (M):

From the figure of shear force:



$$\text{S.F. at point M} = 0 \text{ KN}$$

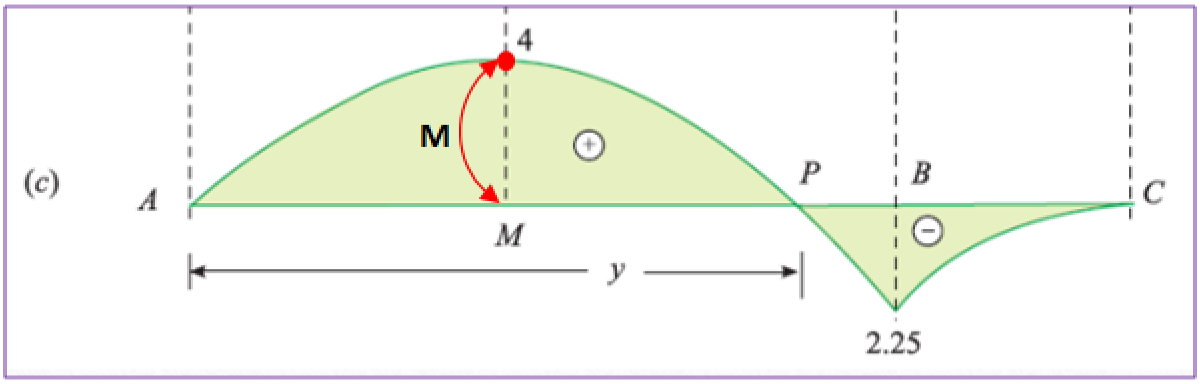
$$\frac{\text{x}}{6} = \frac{3 - x}{7.5}$$

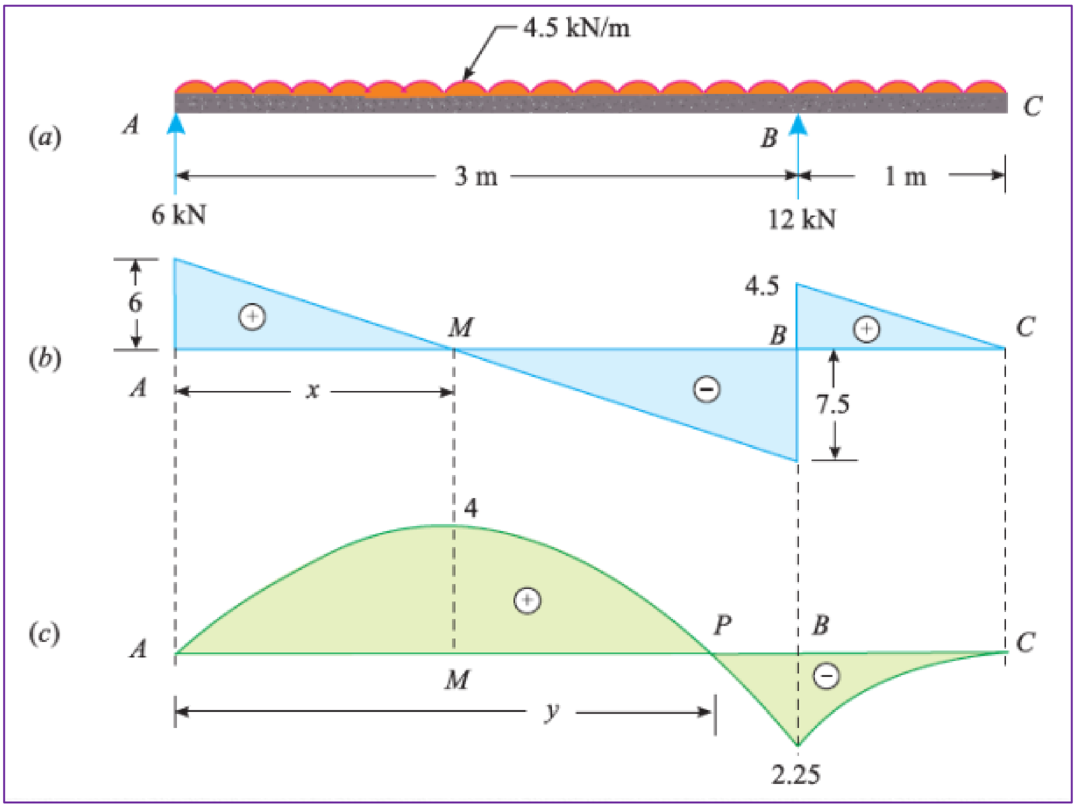
$$7.5 \text{ x} = 18 - 6\,x$$

$$\text{x} = \frac{18}{13.5} = 1.33\ m$$

The bending moment: B.M. at point M

$$B.M.\ at\ point\ B = 6 \times 1.33 - 4.5 \times 1.33 \times \frac{1.33}{2} = 4 \text{ KN . m}$$





To locate the point (p) of contra flexure:

$$\Sigma M_P = 0$$

$$R_A . \mathrm{y} - 4.5 \times \mathrm{y}.\ \frac{y}{2} = 0$$

$$6\mathrm{y} - 2.25\ \mathrm{y}^2 = 0$$

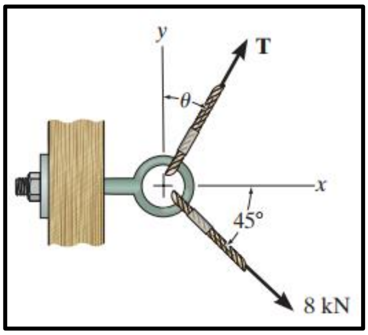
$$\mathrm{y}\ (6 - 2.25\ \mathrm{y}) = 0$$

$$\mathrm{y} = \frac{6}{2.25} = 2.67\ m$$

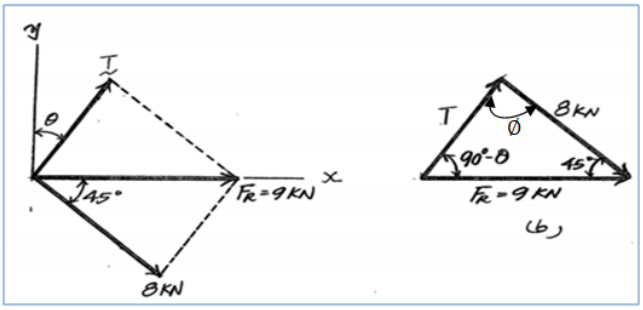
Solve Question Home Work

1.7. Chapter Questions

8. If the magnitude of the resultant force is to be (9 KN) directed along the positive x - axis, determine the magnitude of force (T) acting on the eyebolt and its angle.



Solution



The parallelogram law of addition and the triangular rule are shown in figures (a & b), respectively.

$$R = \sqrt{F_1^2 + F_2^2 - 2F_1 . F_2 \cos\theta}$$

$$T = \sqrt{8^2 + 9^2 - 2 \times 8 \times 9 \times \cos 45} = 6.57\ KN$$

Appling the law of sine's to figure b. and using this result yield:

$$\frac{R}{sin\theta} = \frac{F_1}{sin\beta} = \frac{F_2}{sin\alpha}$$

$$\frac{6.57}{sin45^o} = \frac{8}{sin(90^o - \theta)} = \frac{9}{sin\emptyset}$$

$$\frac{6.57}{sin45^o} = \frac{9}{sin\emptyset}$$

$$sin\emptyset = \frac{9\ sin45^o}{6.57} = 0.968$$

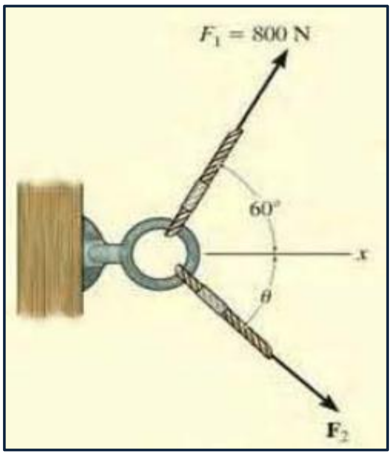
$$\emptyset = sin^{-1}(0.968) = 75.47^o$$

$$90^o - \theta = 180 - 75.47 - 45$$

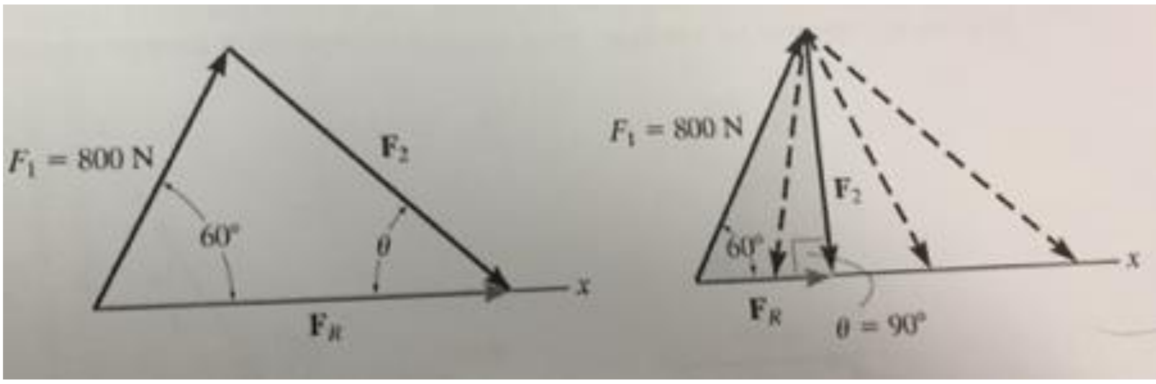
$$90^o - \theta = 119.5$$

$$\theta = 119.5 - 90 = 29.5^o$$

9. It is required that the resultant force acting on the eyebolt in Figure be directed along the positive axis and that (F2) have a minimum magnitude. Determine this magnitude. the angle (θ), and the corresponding resultant force.



Solution



F_2 is a minimum or the shortest length when its line of action is perpendicular to the line of action of F_R, that is, when: $\theta = 90^{\circ}$

$$\frac{800}{sin90^o} = \frac{F_R}{sin30^o} = \frac{F_2}{sin60^o}$$

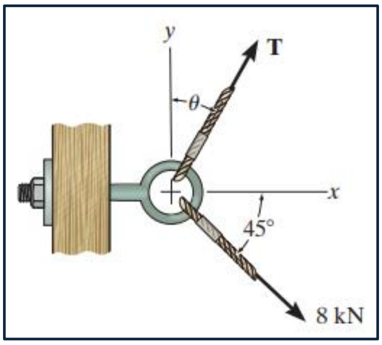
$$\frac{800}{1} = \frac{F_R}{0.5} = \frac{F_2}{0.866}$$

$$F_R = 800 \times 0.5 = 400\ N$$

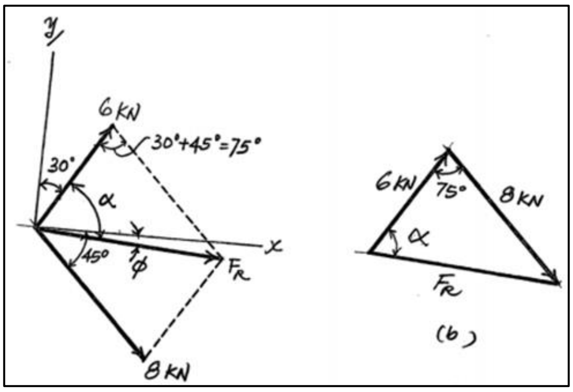
$$F_2 = 800 \times 0.866 = 692.8\ N$$

{Results: $\theta = 90^{\circ}$; $F_R = 400\ N$; $F_2 = 693\ N$}

10. If ($\theta = 30^{\circ}$) and ($\mathrm{T} = 6$ KN) , determine the magnitude of the resultant force acting on the eyebolt and its direction measured clockwise from the positive x axis (∅).



Solution



The parallelogram law of addition and the triangular rule are shown in figures (a & b), respectively.

$$\mathrm{R} = \sqrt{\mathrm{F}_1^2 + \mathrm{F}_2^2 - 2\mathrm{F}_1 . \mathrm{F}_2 \cos\theta}$$

$$\mathrm{R} = \sqrt{6^2 + 8^2 - 2 \times 6 \times 8 \times \cos 75^{o}} = 8.67\ KN$$

Appling the law of sine's to figure b. and using this result yield:

$$\frac{R}{sin\theta} = \frac{F_1}{sin\beta} = \frac{F_2}{sin\alpha}$$

$$\frac{8.67}{sin75^{o}} = \frac{8}{sin\alpha} = \frac{6}{sin\beta}$$

$$sin\alpha = \frac{8\ sin75^{o}}{8.67} = 0.891$$

$$\alpha = \sin^{-1}(0.891) = 63^{o}$$

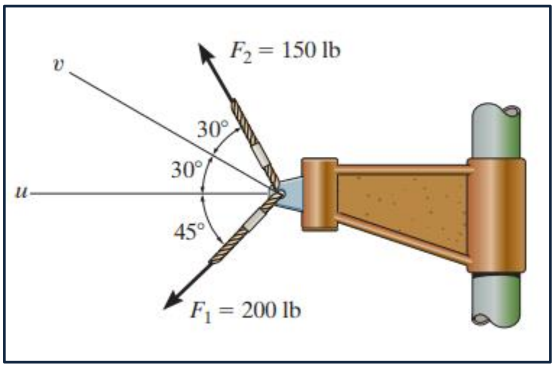
$$sin\beta = \frac{6\ sin75^{o}}{8.67} = 0.668$$

$$\beta = \sin^{-1}(0.668) = 42^{o}$$

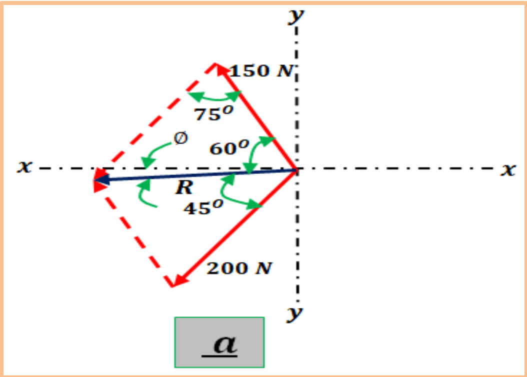
$$\emptyset = \alpha - 60^{o} = 63^{o} - 60^{o} = 3^{o}$$

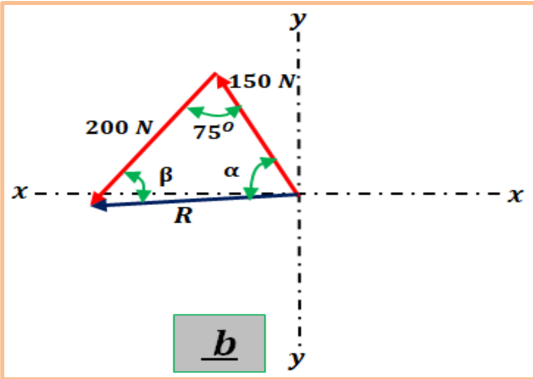
{*Results:* $F_R = 8.67\ KN$; $\alpha = 63.05^{\circ}$; $\emptyset = 3.05^{\circ}$}

11. Determine the magnitude of the resultant force acting on the bracket and its direction measured counterclockwise from the positive u axis.



Solution





The parallelogram law of addition and the triangular rule are shown in figures (a & b), respectively.

$$R = \sqrt{F_1^2 + F_2^2 - 2F_1 . F_2 \cos\theta}$$

$$R = \sqrt{200^2 + 150^2 - 2 \times 200 \times 150 \times \cos 75^O} = 216.72\ N$$

Appling the law of sine's to figure b. and using this result yield:

$$\frac{R}{\sin\theta} = \frac{F_1}{\sin\beta} = \frac{F_2}{\sin\alpha}$$

$$\frac{216.72}{\sin 75^o} = \frac{200}{\sin\alpha} = \frac{150}{\sin\beta}$$

$$\sin\alpha = \frac{200 \sin 75^o}{216.72} = 0.891$$

$$\alpha = \sin^{-1}(0.891) = 63^o$$

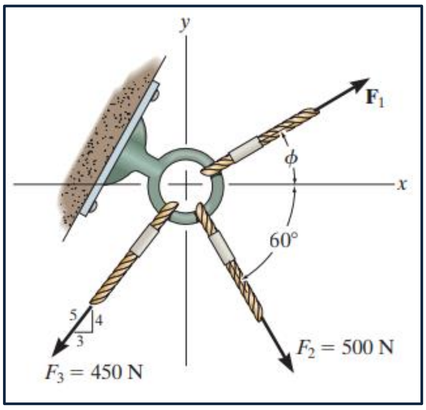
$$\sin\beta = \frac{150 \sin 75^o}{216.72} = 0.668$$

$$\beta = \sin^{-1}(0.668) = 42^o$$

$$\emptyset = \alpha - 60^o = 63^o - 60^o = 3^o$$

{*Results:* $F_R = 217\ N\ ; \alpha = 63.05^{\circ}\ ; \emptyset = 3.05^{\circ}$}

12. If $(F_1 = 600\ N)$ and $(\emptyset = 30^{\circ})$, determine the magnitude of the resultant force acting on the eyebolt and its direction measured clockwise from the positive x axis



Solution

$$\theta = \tan^{-1}\frac{3}{4} = 36.87^{o}$$

NO.	***Description***	$\boldsymbol{\Sigma F_x}$ ***(N)***	$\boldsymbol{\Sigma F_y}$ ***(N)***
1.	$600 \llcorner 30^{o}$	$600\cos 30^{o} = 519.6$	$600\sin 30^{o} = 300$
2.	$450 \llcorner 233.13^{o}$	$450\cos 233.13^{o} = -270$	$450\sin 233.13^{o} = -353.83$
3.	$500 \llcorner 300^{o}$	$500\cos 300^{o} = 250$	$500\sin 300^{o} = -433.01$
Sum		**499.6**	**-486.84**

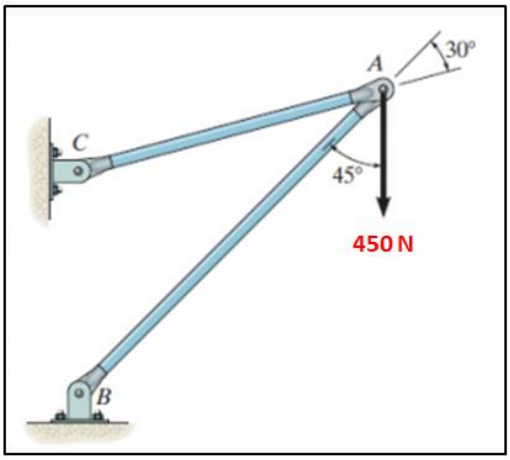
$$\boldsymbol{R} = \sqrt{\boldsymbol{\Sigma F_x^2} + \boldsymbol{\Sigma F_y^2}} = \sqrt{(499.6)^2 + (-486.84)^2} = \mathbf{697.58\ N}$$

$$\boldsymbol{\theta} = \mathbf{tan^{-1}}\left(\frac{\boldsymbol{F_y}}{\boldsymbol{F_x}}\right) = \tan^{-1}\left(\frac{486.84}{499.6}\right) = \mathbf{44.26}^{\circ}$$

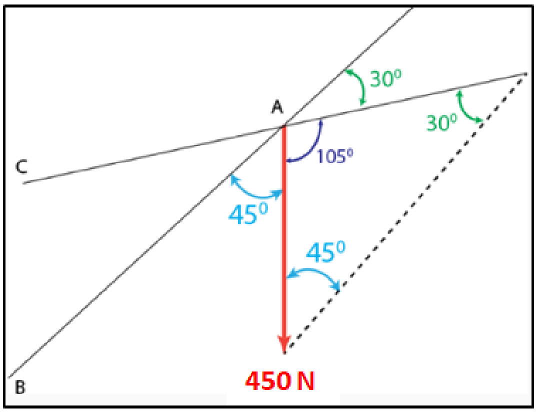
{Results: $F_R = 701.91\ N\ ; \theta = 44.06^{\circ}$ }

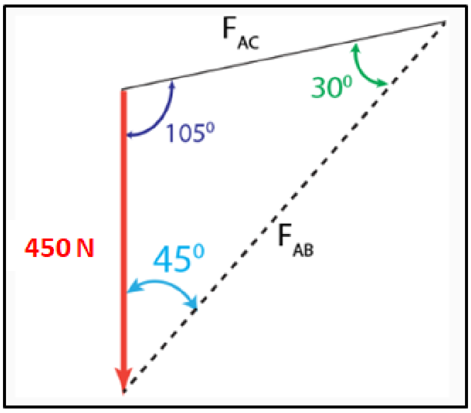
---

13. The force (F = 450 N) acts on the frame. Resolve this force into components acting along members AB and AC , and determine the magnitude of each component.



Solution





Appling the law of sine's to find (F_{AB} & F_{AC}):

$$\frac{R}{sin\theta} = \frac{F_{AB}}{sin\alpha} = \frac{F_{AC}}{sin\beta}$$

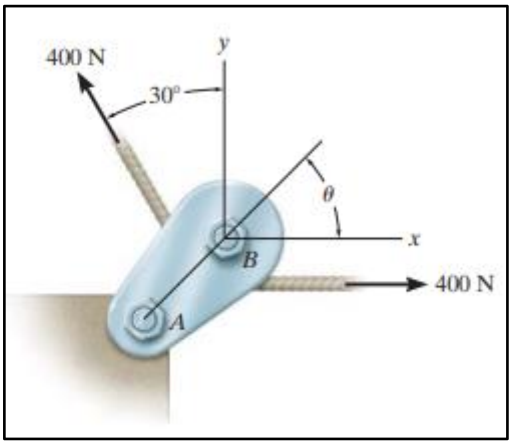
$$\frac{450}{sin30^o} = \frac{F_{AB}}{sin105^o} = \frac{F_{AC}}{sin45^o}$$

$$F_{AB} = \frac{450\ sin105^o}{sin30^o} = 869.33\ N$$

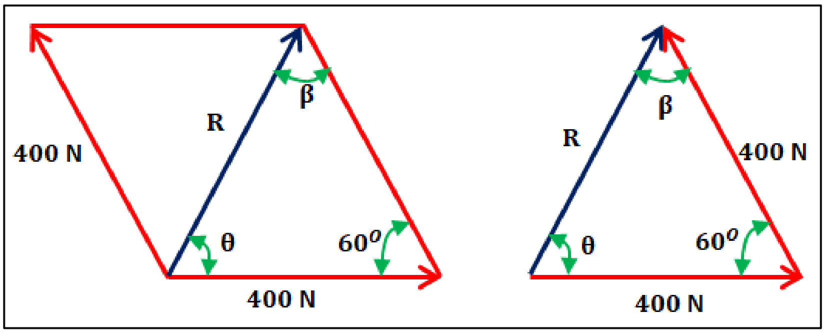
$$F_{AC} = \frac{450\ sin45^o}{sin30^o} = 636.4\ N$$

{Results: $F_{AB} = 869\ N\ ; F_{AC} = 636\ N$ }

14. If the tension in the cable is 400 N, determine the magnitude and direction of the resultant force acting on the pulley. This angle is the same angle of line AB on the tailboard block.



Solution



$$R = \sqrt{400^2 + 400^2 - 2 \times 400 \times 400 \times \cos(60^{\circ})} = 400\ N$$

From sine law:

$$\frac{R}{sin\alpha} = \frac{F_{AB}}{sin\theta} = \frac{F_{AC}}{sin\beta}$$

$$\frac{400}{\sin 60^o} = \frac{400}{sin\ \theta} = \frac{400}{sin\ \beta}$$

$$\sin \theta^o = \frac{400 sin\ 60^o}{400} = 0.866$$

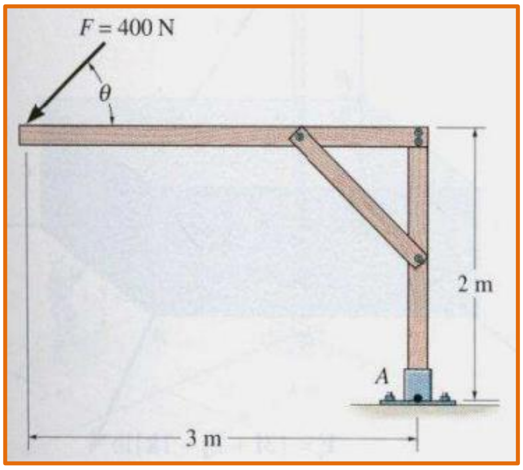
$$\theta^o = 60^o$$

$$\sin \beta = \frac{400 sin\ 60^o}{400} = 0.866$$

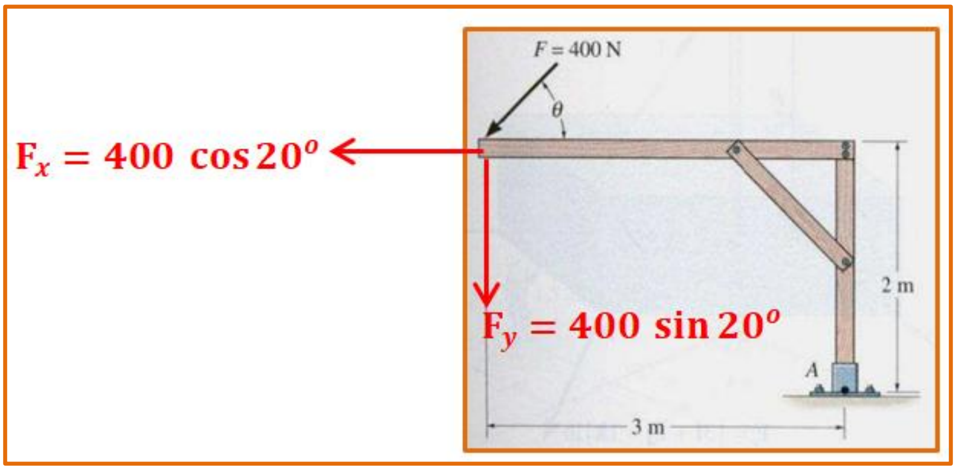
$$\beta = 60^o$$

2. 7. Chapter Questions

1. A 400 N force is applied to the frame and ϴ = 20º, as in the following figure. Find the moment of the force at A?



Solution:



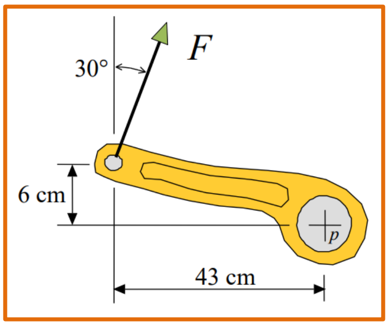
$$\circlearrowright +\mathbf{M_B} = \mathbf{\Sigma\, F\,.\,d}$$

$$\circlearrowright + \mathrm{M_B} = -400\cos 20^{o} \times 2 - 400\sin 20^{o} \times 3 = -751.75 - 410.42$$

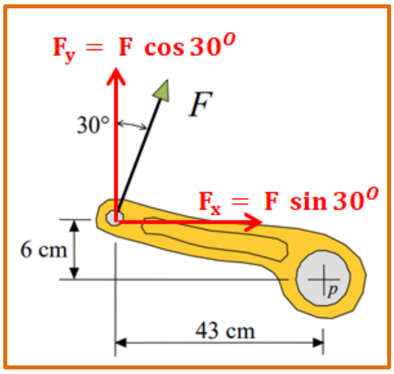
$$= -1162.17\ N\,.\,m = 1162.17\ N\,.\,m\ \circlearrowleft$$

{*Answer:* $M_B = 1160\ N.m$}

2. The wrench shown is used to turn drilling pipe. If a torque (moment) of (800 N.m) about point (p) is needed to turn the pipe, determine the required force (F).



Solution:



$$\circlearrowright +\mathbf{M_p} = \mathbf{\Sigma F.d}$$

$$\circlearrowright + M_p = F\sin 30^o \times 0.06 + F\cos 30^o \times 0.43$$

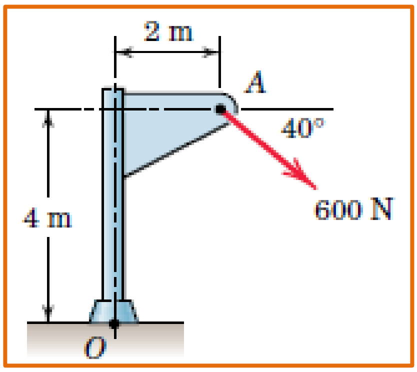
$$800 = 0.03\,F + 0.372\,F$$

$$F = \frac{800}{0.402} = 1990\ N$$

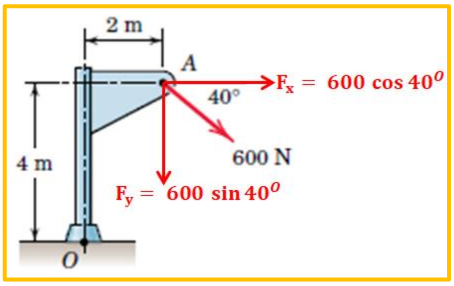
{Answer: $F = \mathbf{1990}\ N$}

---

3. Calculate the moment about the base point (O) of the (600 N)?



Solution:



$$\circlearrowright +\mathbf{M_A} = \mathbf{\Sigma\, F\,.\,d}$$

$$\circlearrowright + M_O = 600\,.\cos40^o \times 4 + 600.\sin40^o \times 2$$

$$\circlearrowright + M_O = 1838.51 \; + 771.35 = \; 2609.86\ N.m \circlearrowright$$

{*Answer:* $\boldsymbol{M_o = 2610\ N.m}$}

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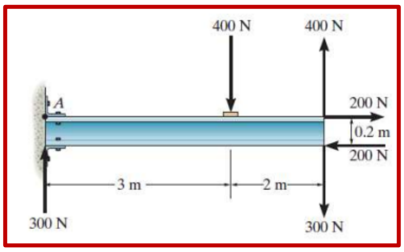
4. Determine the resultant moment acting on the beam?



Solution:

$$\circlearrowright +\mathbf{M_A} = \mathbf{\Sigma\, F\,.\,d}$$

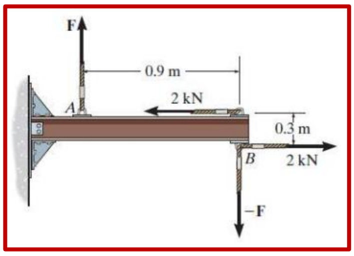
$$\circlearrowright + M_A = -400 \times 2 + 300 \times 5 + 200 \times 0.2$$

$$\circlearrowright + M_A = 800 + 1500 + 40 = \; 740\ N.m \circlearrowright$$

{*Answer:* $M_{Couple} = 740\ N.m\ \circlearrowright$}

---

5. Determine the magnitude of (F), so that the resultant moment acting on the beam is (1.5 kN.m) clockwise.?



Solution:

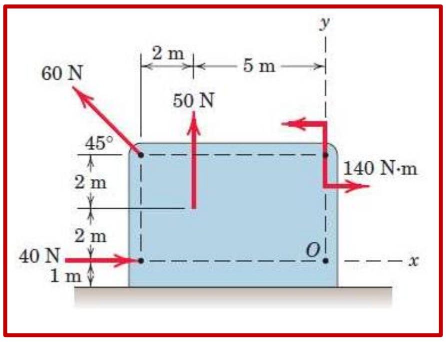
$$\circlearrowright +\mathbf{M_A} = \mathbf{\Sigma\, F\,.\,d}$$

$$1.5 = F \times 0.9 - 2 \times 0.3$$

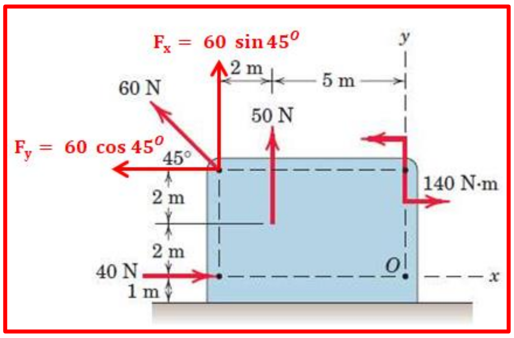
$$F = \frac{1.5 + 0.6}{0.9} = \; 2.333\ KN$$

{*Answer:* $F \; = \; 2.333\ KN$}

6. Determine the resultant moment of the three forces and one couple which act on the plate shown about point (O)?



Solution:



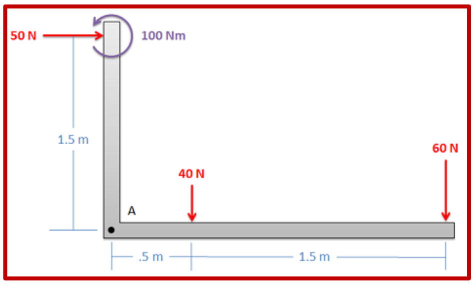
$$\circlearrowright +\mathbf{M_A} = \mathbf{\Sigma F . d}$$

$$\circlearrowright + M_O = -140 + 50 \times 5 - 60 . \cos 45^o \times 4 + 60 . \sin 45^o \times 7 - 40 \times 0$$

$$\circlearrowright + M_O = -140 + 250 - 169.7 + 296.98 - 0 = 237.28\ N.m \circlearrowright$$

{Answer:: $M_O = 237.28\ N.m\ \circlearrowright$}

7. Find the equivalent force couple system about point A for the set of forces shown below?



Solution:

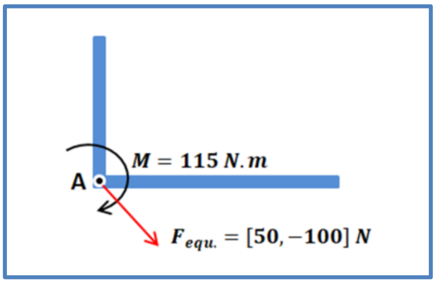
$\mathbf{\Sigma F_x} = 50\ N$

$\mathbf{\Sigma F_y} = -40 - 60 = -100\ N$

$\circlearrowright +\mathbf{M_A} = \mathbf{\Sigma F . d}$

$\circlearrowright + M_A = 500 \times 1.5 - 100 + 40 \times 0.5 + 60 \times 2$

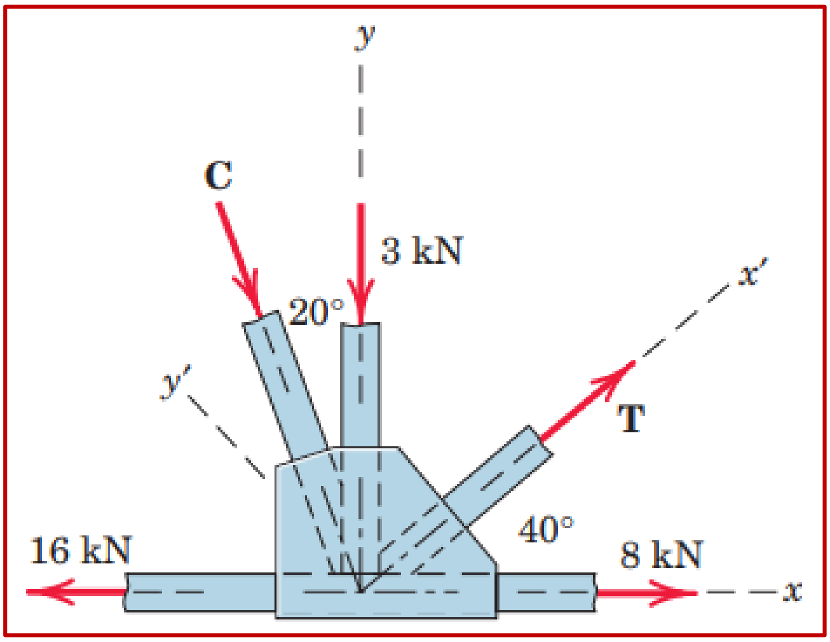
$\circlearrowright + M_A = 75 - 100 + 20 + 120 = 115\ N.m \circlearrowright$



{Answer:: $M_A = 115\ N.m\ \circlearrowleft$}

3.6. Chapter Questions

1. Determine the magnitudes of the forces C and T, which, along with the other three forces shown, act on the bridge-truss joint?



Solution I (scalar algebra). For the x-y axes as shown we have

$$[\Sigma F_x = 0] \qquad 8 + T\cos 40° + C\sin 20° - 16 = 0$$

$$0.766T + 0.342C = 8$$

$$[\Sigma F_y = 0] \qquad T\sin 40° - C\cos 20° - 3 = 0$$

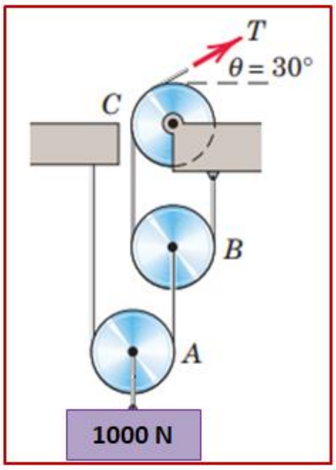
$$0.643T - 0.940C = 3$$

Simultaneous solution of Eqs. (a) and (b) produces

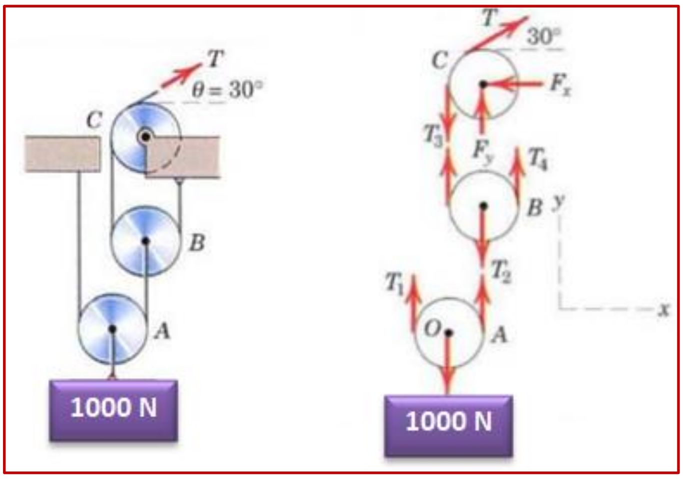
$$T = 9.09 \text{ kN} \qquad C = 3.03 \text{ kN}$$

{*Answer:* $T = 9.09\ KN$, $C = 3.03\ KN$}

2. Determine the magnitudes of the forces C and T, which, along with the other three forces shown, act on the bridge-truss joint?



Solution:



$$M_o = 0$$

$$T_1 \, . \, r - T_2 \, . r = 0$$

$$\Sigma F_y = 0$$

$$T_1 + T_2 - 1000 = 0$$

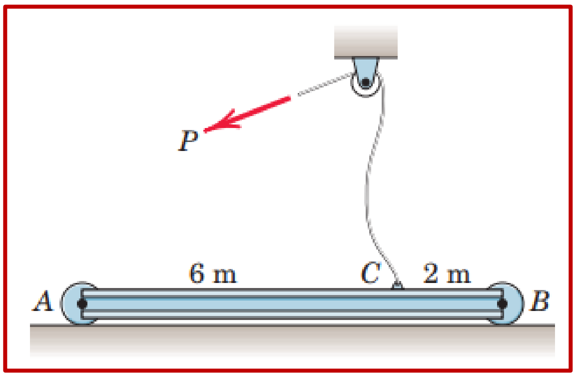
$$T_1 = T_2 = 500 \ N$$

$$T_3 = T_4 = \frac{T_2}{2} = 500 \ N$$

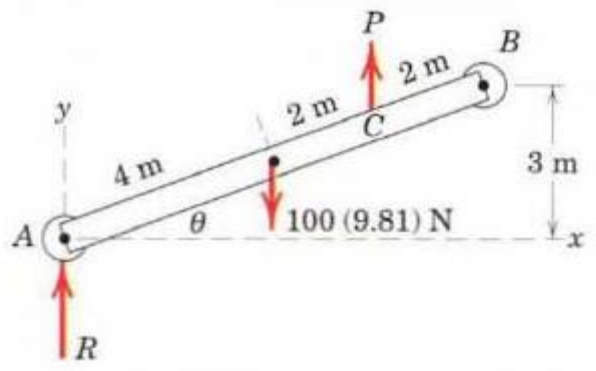
$$T = T_3 = 500 \ N$$

{Answer: $\boldsymbol{F = 250 \ N}$}

3. The uniform (100 kg) I-beam is supported initially by its end rollers on the horizontal surface at A and B. By means of the cable at C it is desired to elevate end B to a position (3 m) above end A. Determine the required tension P, the reaction at A, and the angle made by the beam with the horizontal in the elevated position.



Solution



Moment equilibrium about A eliminates force *R* and gives

$\Sigma M_o = 0$ $\qquad$ $P(6 \cos\theta) - 981(4 \cos\theta) = 0$ $\qquad$ $P = 654$ N

Equilibrium of vertical forces requires

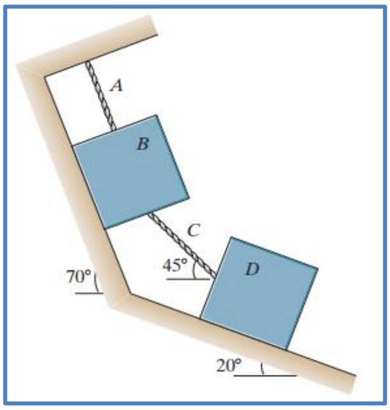
$\Sigma F_y = 0$ $\qquad$ $654 + R - 981 = 0$ $\qquad$ $R = 327$ N

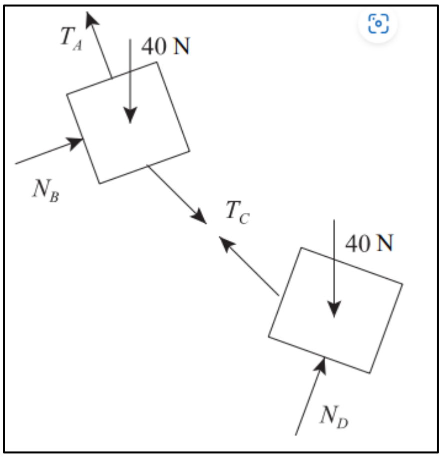
The angle θ depends only on the specified geometry and is

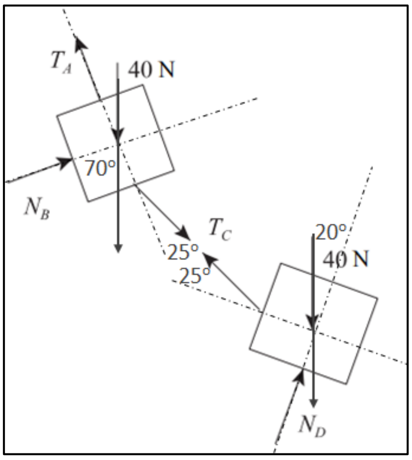
$\sin\theta = 3/8$ $\qquad$ $\theta = 22^\circ$

{Answer: $P = 654\ N\ ,\ \theta = 22^o$}

4. Each box weighs 40 N. The angles are measured relative to the horizontal. The surfaces are smooth. Determine the tension in the rope A and the normal force exerted on box B by the inclined surface?







The free-body diagrams are shown. The equilibrium equations for box D are

$$\sum F_x : (40\ \text{N}) \sin 20^\circ - T_C \cos 25^\circ = 0$$

$$\sum F_y : N_D - (40\ \text{N}) \cos 20^\circ + T_C \sin 25^\circ = 0$$

The equilibrium equations for box B are

$$\sum F_x : (40\ \text{N}) \sin 70^\circ + T_C \cos 25^\circ - T_A = 0$$

$$\sum F_y : N_B - (40\ \text{N}) \cos 70^\circ + T_C \sin 25^\circ = 0$$

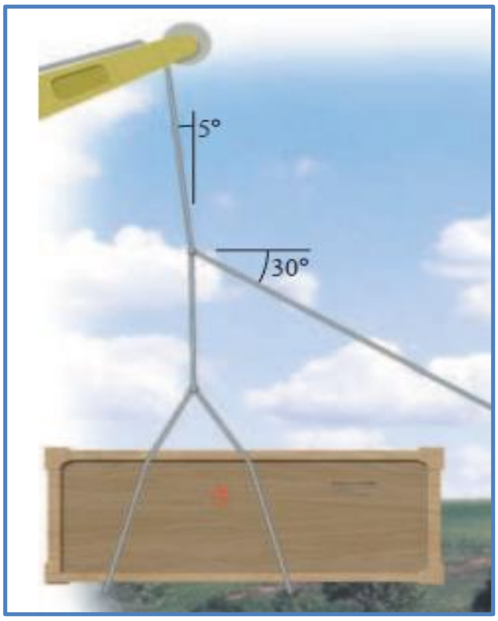
Solving these four equations yields:

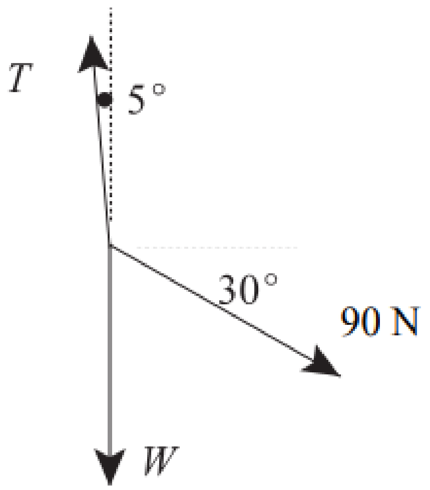
T_A = 51.2 N, T_C = 15.1 N, N_B = 7.30 N, N_D = 31.2 N

Thus **T_A = 51.2 N, NB = 7.30 N.**

{*Answer:* $T_A = 51.2\ N$, $N_B = 7.03\ N$}

5. The construction worker exerts a 90 N force on the rope to hold the crate in equilibrium in the position shown. What is the weight of the crate?





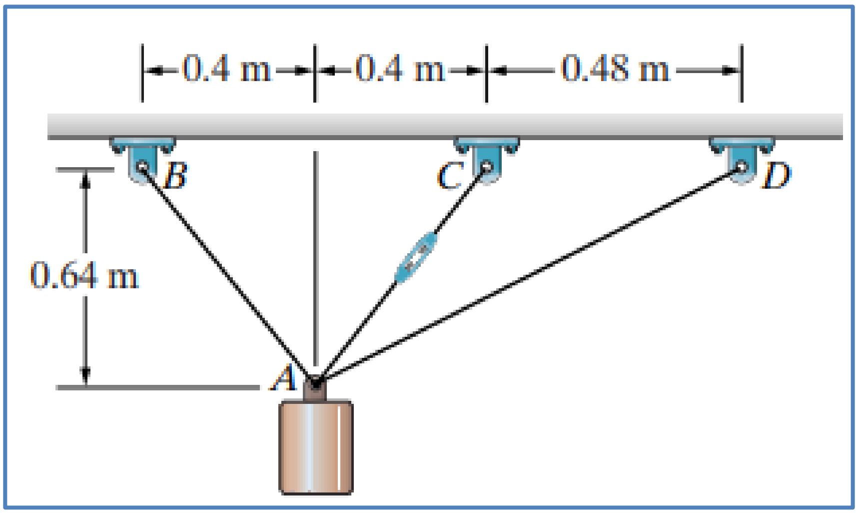
The free-body diagram is shown. The equilibrium equations for the part of the rope system where the three ropes are joined are

$$\sum F_x : (90\ \text{N}) \cos 30° - T \sin 5° = 0$$
$$\sum F_y : -(90\ \text{N}) \sin 30° + T \cos 5° - W = 0$$

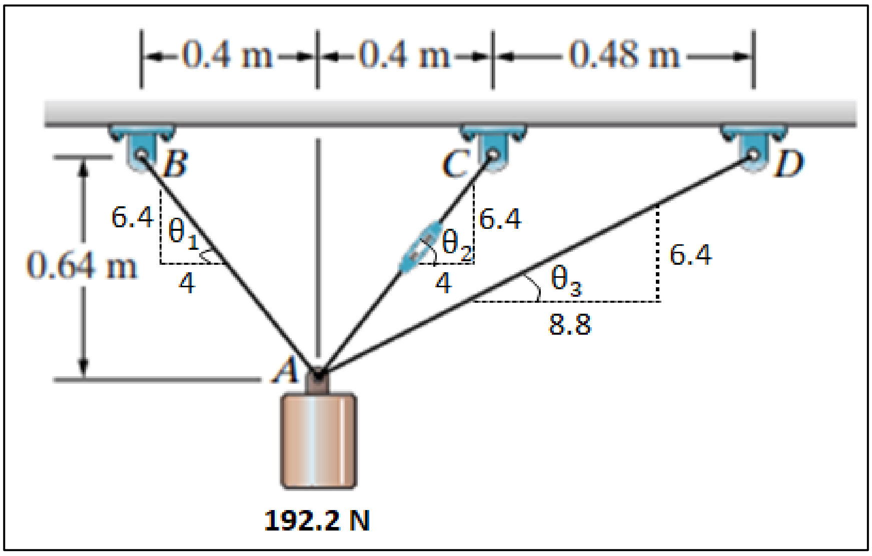
Solving yields **W = 935.9 N.**

{Answer: $W = 935.9\ N$*}*

6. The 100 kg mass is suspended from three cables. Cable AC is equipped with a turnbuckle so that its tension can be adjusted and a strain gauge that allows its tension to be measured. If the tension in cable AB is 200 N, what are the tensions in cables AB and AD?



Solution:



$$T_{Ac} = 40\ N$$

$$\theta_1 = tan^{-1}\left(\frac{6.4}{4}\right) = 57.99^o$$

$$\theta_2 = tan^{-1}\left(\frac{6.4}{4}\right) = 57.99^o$$

$$\theta_3 = tan^{-1}\left(\frac{6.4}{8.8}\right) = 36.03^o$$

$$\Sigma F_x = 0$$

$$\sin 57.99^o = 0.848 \quad ; \quad cos\, 57.99^o = 0.531$$

$$\sin 36.03^o = 0.588 \quad ; \quad cos\, 36.03^o = 0.809$$

$$-T_{AB}.cos\theta_1 + T_{Ac}.cos\theta_2 + T_{AD}.cos\theta_3 = 0$$

$$-106 + 0.53\, T_{AC} + 0.81\, T_{AD} = 0$$

$$T_{AC} = \frac{106 - 0.81\, T_{AD}}{0.53} \qquad (1)$$

$$\Sigma F_y = 0$$

$$T_{AB}.sin\theta_1 + T_{Ac}.sin\theta_2 + T_{AD}.sin\theta_3 - 981 = 0$$

$$169.6 + 0.848\, T_{AC} + 0.588\, T_{AD} - 981 = 0 \qquad (2)$$

Substituting the first equation with the second equation results:

$$169.6 + 0.848\, (\frac{106 - 0.81\, T_{AD}}{0.53}) + 0.588\, T_{AD} - 981 = 0 \qquad (2)$$

$$169.6 + 1.6\, (106 - 0.81\, T_{AD})\ + 0.588\, T_{AD} - 981 = 0$$

$$169.6 + 169.6 - 0.81\, T_{AD}\ + 0.588\, T_{AD} - 981 = 0$$

$$1.6\ (21.2 + 0.81\, T_{AD}) + 0.588\, T_{AD} - 196.2 = 0$$

$$33.92 + 1.296\, T_{AD} + 21.2 + 0.81\, T_{AD} - 196.2 = 0$$

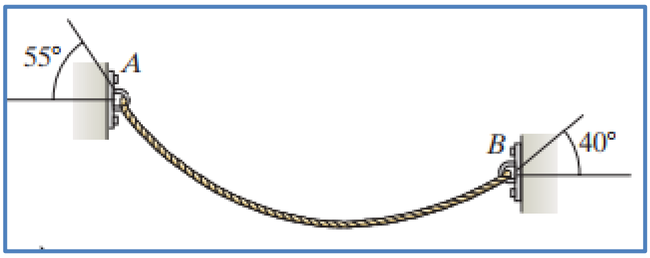
$$T_{AD} = \frac{128.36}{1.884} = 68.13\, N$$

Substituting the value of (T_{AD}) into the equation (I) to obtain the value of (T_{AB}) results in:

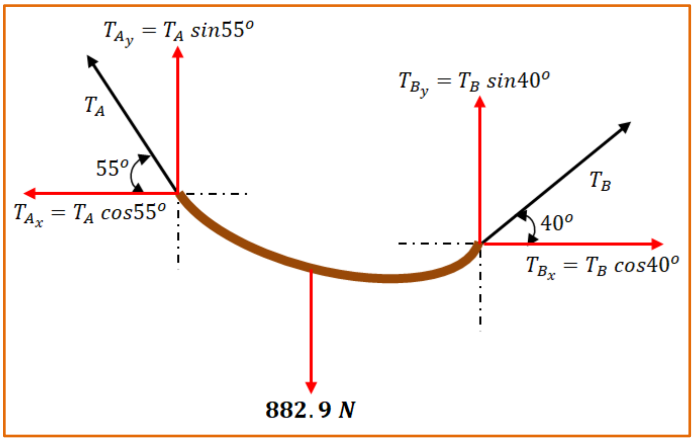
$$T_{AB} = \frac{21.2 + 0.81(68.13)}{0.53} = 144.14\, N$$

{Answer: $\boldsymbol{T_{AB} = 144.1\, N}$, $\boldsymbol{T_{AD} = 68.2\, N}$}

7. A heavy rope used as a mooring line for a cruise ship sags as shown. If the mass of the rope is 90 kg, what are the tensions in the rope at A and B?



Solution:



$$\Sigma F_x = 0$$

$$T_B\,cos40^o - T_A\,cos55^o = 0$$

$$0.766\ \mathrm{T_B} - 0.575\ \mathrm{T_A} = 0$$

$$\mathrm{T_B} = \frac{0.575\ T_A}{0.766}$$

$$\mathrm{T_B} = 0.751\ \mathrm{T_A} \qquad (1)$$

$$\Sigma F_y = 0$$

$$T_B\,sin40^o + T_A\,sin55^o - 882.9 = 0$$

$$0.643\ T_B + 0.819\ T_A - 882.9 = 0 \qquad (2)$$

Substituting the first equation with the second equation results:

$$0.643\ (0.751\ T_A) + 0.819\ T_A - 882.9 = 0$$

$$\mathrm{T_A} = \frac{882.9}{1.302} = 678.11\ N$$

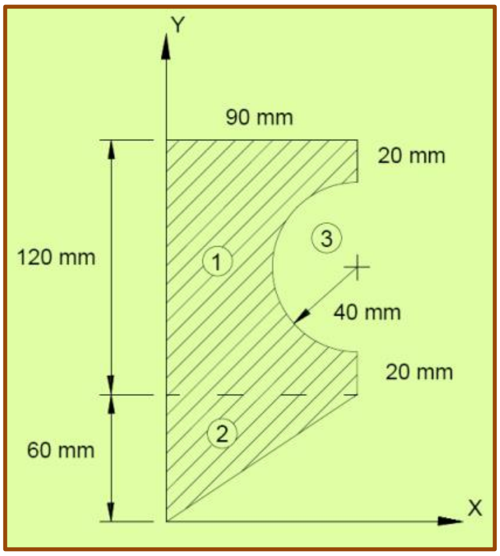
Substituting the value of (T_A) into the equation (1) to obtain the value of (T_B) results in:

$$\mathrm{T_B} = 0.751\ (678.11) = 509.26\ \mathrm{N}$$

{Answer: $\boldsymbol{T_A = 679\ N\ ,\ \ T_B = 508\ N}$**}**

4.8 Chapter Questions

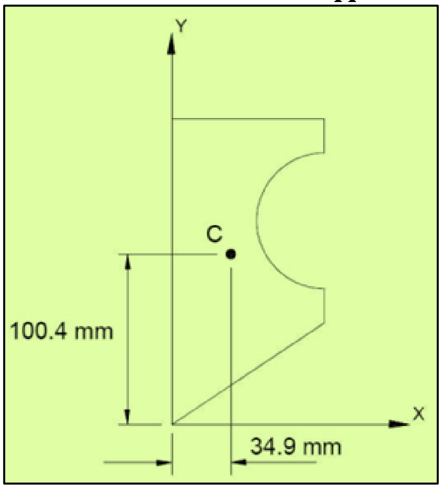
Q1: Locate the centroid of the area shown in the figure below?



Solution

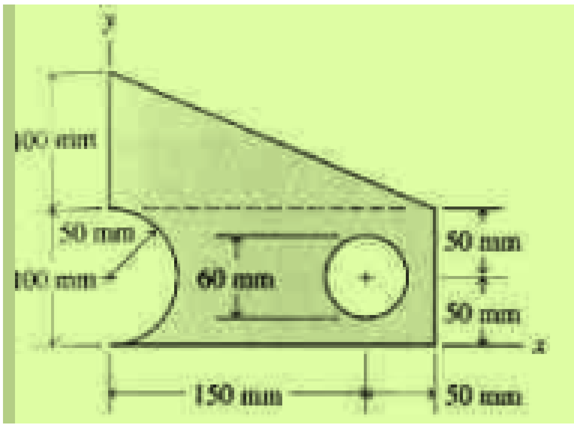
Part	Area, A_i	$\bar{x}_i$	$\bar{y}_i$	$\bar{x}_i A_i$	$\bar{y}_i A_i$
1	10,800	45.0	120.0	486,000	1,296,000
2	2,700	30.0	40.0	81,000	108,000
3	- 2,510	73.0	120.0	- 183,000	- 301,000
Totals	10,990			384,000	1,103,000

$$x_C = \frac{\Sigma x_i . A_i}{A} = \frac{384000}{10990} = 34.94\ mm\ , \qquad y_C = \frac{\Sigma y_i . A_i}{A} = \frac{110300}{10990} = 100.4\ mm$$

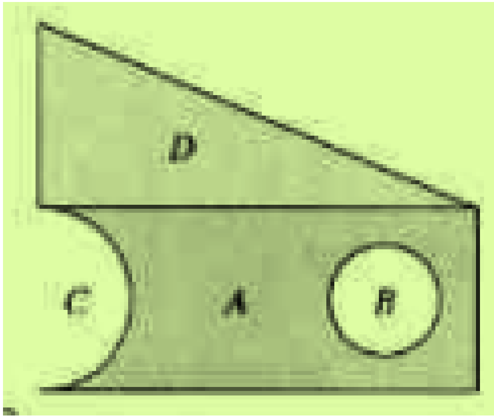


{Answer: $x_C = 34.9\ mm\ , and\ y_C = 100.4\ mm$}

Q_2: Locate the centroid of the area shown in the figure below?



Solution



shape	A (mm²)	x̄ (mm)	x̄A (mm³)	ȳ (mm)	ȳA (mm³)
A	20000	100	2000000	50	1000000
B	-2827.43	150	-424115	50	-141372
C	-3926.99	21.22066	-83333.3	50	-196350
D	10000	66.66667	666666.7	133.3333	1333333
	23245.58		2159218		1995612

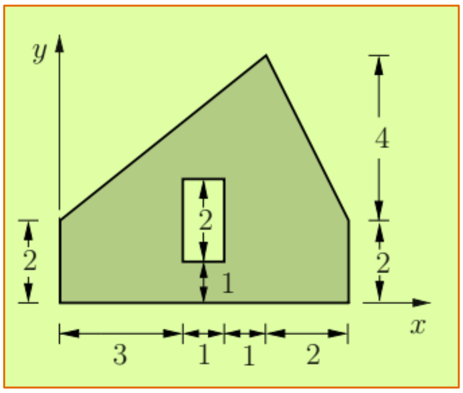
$$\hat{x} = \frac{2159218\,mm^3}{23245.58\,mm^2} = 92.9\ mm$$

$$\hat{y} = \frac{1995612\ mm^3}{23245.58\ mm^2} = 85.8\ mm$$

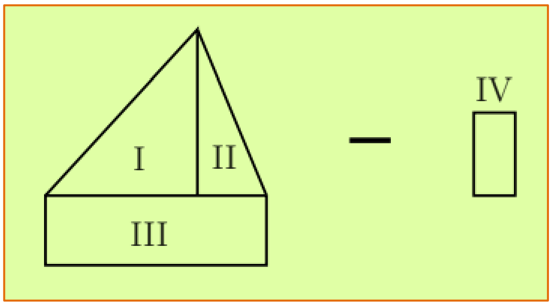
$$x_C = \frac{\Sigma x_i . A_i}{A} = \frac{2159218}{23245.58} = 92.89\ mm\ ,\ y_C = \frac{\Sigma y_i . A_i}{A} = \frac{1995612}{23245.58} = 85.85\ mm$$

{Answer: $x_C = 92.9\ mm\ , and\ y_C = 85.8\ mm$}

Q$_3$: Locate the centroid of the depicted area with a rectangular cutout. The measurements are given in meter?



Solution



The calculation is conveniently done by using a table.

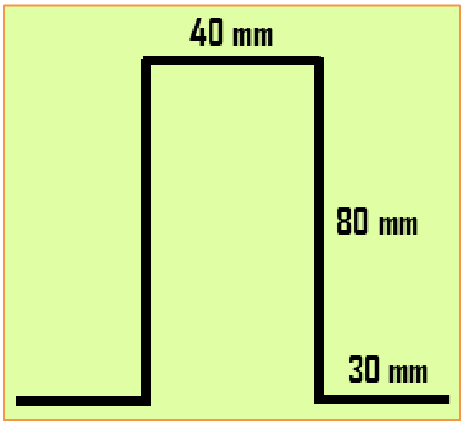
Segment	$A_i\ (m^2)$	$x_i\ (m)$	$x_i\ .A_i\ (m^3)$	$y_i\ (m)$	$y_i\ .A_i\ (m^3)$
I	10	3.33	33.3	3.33	33.3
II	4	5.67	22.68	3.33	13.32
III	14	3.5	49	1	14
IV	-2	3.5	-7	2	-4
Sum	**26**		**97.98**		**56.62**

Thus, we obtain:

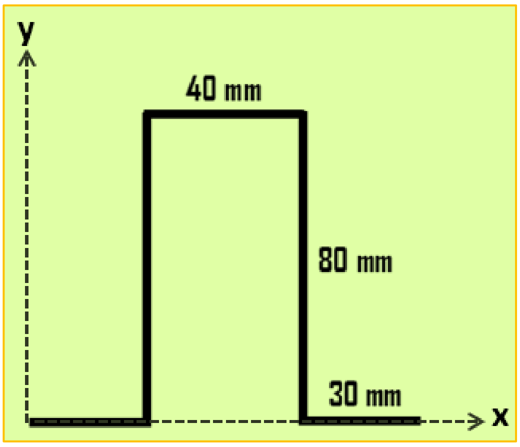
$$x_C = \frac{\Sigma x_i . A_i}{A} = \frac{97.98}{26} = 3.77\ m\ , \quad y_C = \frac{\Sigma y_i . A_i}{A} = \frac{56.62}{26} = 2.18\ m$$

{*Answer:* $x_C = 3.77\ m$, and $y_C = 2.18\ m$ }

Q$_4$: A wire with constant thickness is deformed into the depicted figure. The measurements are given in mm. Find the Locate of the centroid?



Solution

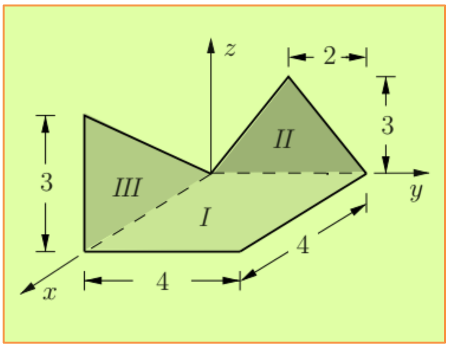


$$x_C = \frac{\Sigma\, x_i\,.L_i}{\Sigma\, L_i} = \frac{30 \times 15 + 80 \times 30 + 40 \times 50 + 80 \times 70 + 30 \times 85}{30 + 80 + 40 + 80 + 30} = 50\ mm$$

$$y_C = \frac{\Sigma\, y_i\,.L_i}{\Sigma\, L_i} = \frac{80 \times 40 + 40 \times 80 + 80 \times 40}{30 + 80 + 40 + 80 + 30} = 36.92\ mm$$

{Answer: $x_C = 50\ mm,\ \ y_C = 36.92\ mm$ }

Q$_5$: A thin sheet with constant thickness and density, consisting of a square and two triangles, is bent to the depicted figure (measurements in meter). Locate the center of gravity?



Solution

The body is composed by three parts with already known location of centers of mass. The location of the center of mass of the complete system can be determined from

$$x_C = \frac{\sum \rho_i x_i V_i}{\sum \rho_i V_i}, \quad y_C = \frac{\sum \rho_i y_i V_i}{\sum \rho_i V_i}, \quad z_C = \frac{\sum \rho_i z_i V_i}{\sum \rho_i V_i}.$$

Since the thickness and the density of the sheet is constant, these terms cancel out and we obtain:

$$x_C = \frac{\sum x_i A_i}{\sum A_i}, \quad y_C = \frac{\sum y_i A_i}{\sum A_i}, \quad z_C = \frac{\sum z_i V_i}{\sum A_i}.$$

The total area is:

$$A = \Sigma A_i = 4 \times 4 + \frac{1}{2} \times 4 \times 3 + \frac{1}{2} \times 4 \times 3 = 28\, m^2$$

Calculating the first area moments of the total system about each axis, in each case one first moment of a subsystem drops out because of zero distance: $x_{II} = 0,\ y_{III} = 0, and\ z_I = 0$. Thus, we obtain:

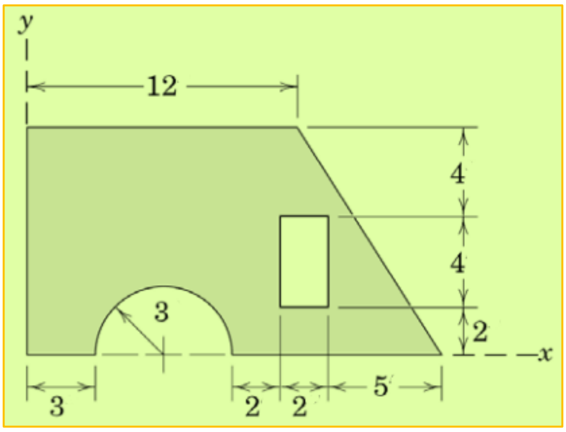
$$x_C = \frac{x_I.A_I + x_{III}.A_{III}}{A} = \frac{2 \times 16 + \left(\frac{2}{3} \times 4\right) \times 6}{28} = 1.71\, m,$$

$$y_C = \frac{y_I.A_I + y.A_{III}}{A} = \frac{2 \times 16 + 2 \times 6}{28} = 1.57\, m$$

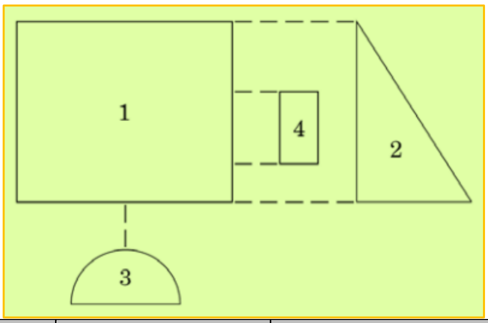
$$z_C = \frac{z_{II}.A_{II} + z_{III}.A_{III}}{A} = \frac{(\frac{1}{3} \times 3) \times 6 + (\frac{1}{3} \times 3) \times 6}{28} = 0.43\, m$$

{*Answer:* $x_C = 1.71\, m,\ y_C = 1.57, and\ z_C = 0.43\, m$ }

Q_6: Locate the centroid of the area shown in the figure below, all dimension in m?



Solution



Segment	$A_i\ (m^2)$	$x_i\ (m)$	$x_i\,.A_i\ (m^3)$	$y_i\ (m)$	$y_i\,.A_i\ (m^3)$
1	120	6	720	5	600
2	30	14	420	3.33	100
3	-14.14	6	-84.8	1.273	-18
4	-8	12	-96	4	-32
Sum	**127.9**		**959**		**650**

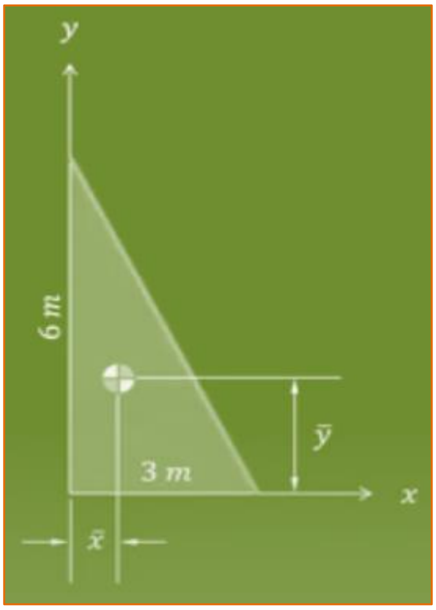
Thus, we obtain:

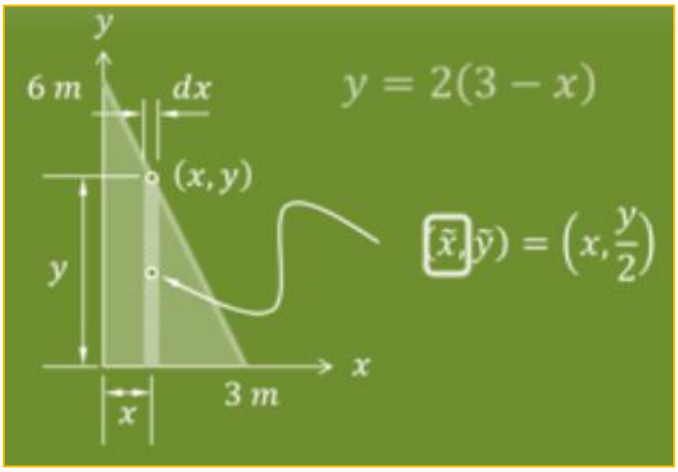
$$x_C = \frac{\Sigma x_i.A_i}{A} = \frac{959}{127.9} = 7.50\ m\,, \quad y_C = \frac{\Sigma y_i.A_i}{A} = \frac{650}{127.9} = 5.08\,m$$

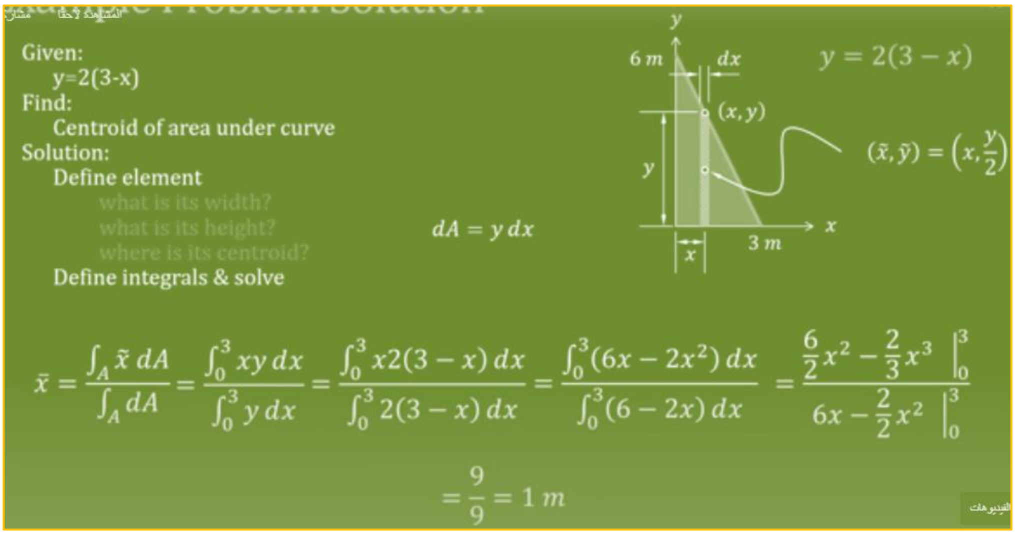
*{**Answer:*** $x_C = 7.5\,m$ *, and* $y_C = 5.08\,m$ }

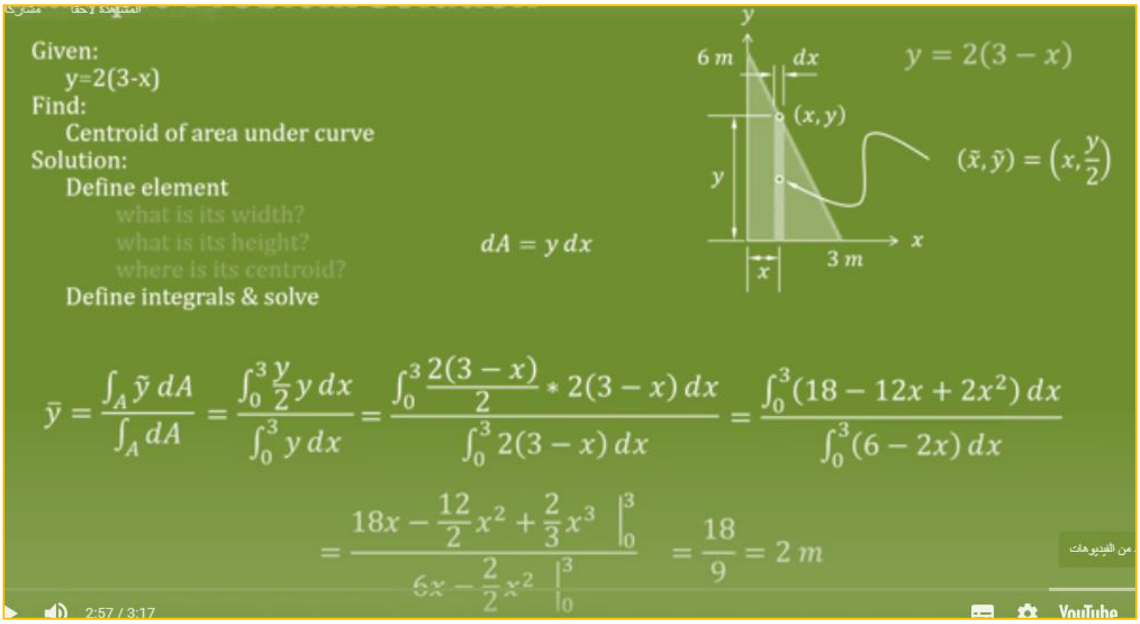
Q$_7$: Find the Locate of the centroid of the area shown in the figure below, by using integration?

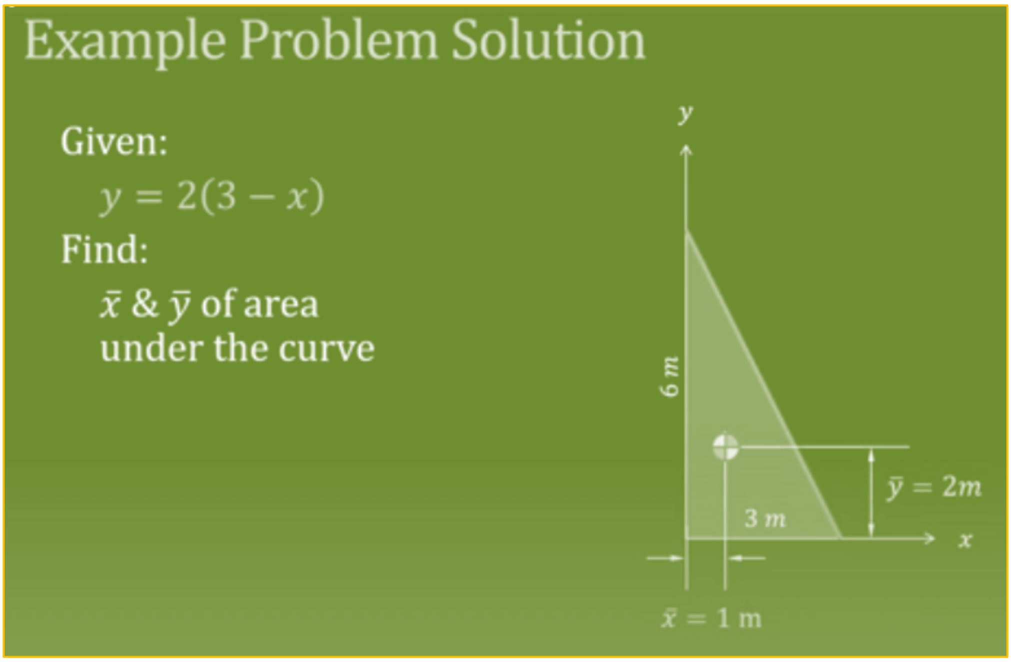
$$y = 2(3 - x)$$







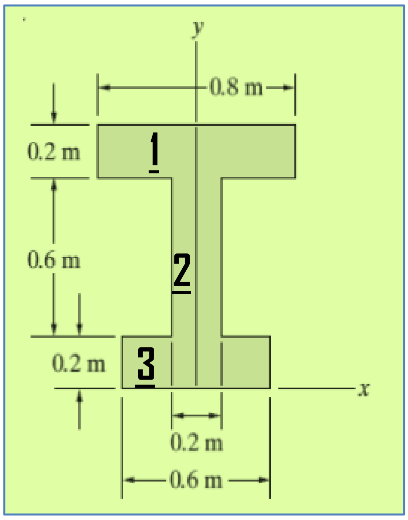




{Answer: $x_C = 1\ m$, $and\ y_C = 2\ m$ }

5. Chapter Questions

Q1: Determine the moment of inertia of the section relative to the x-axis?



Solution:

$$\boldsymbol{A = w\,.\,L}$$

$$\boldsymbol{I_{x_o}\,\frac{1}{12}bh^3\ ;\ k_x = \sqrt{\frac{I_x}{A}}\quad ;\ k_y = \sqrt{\frac{I_y}{A}}}$$

Part	A (m^2)	dx (m)	dy (mm)	A , dx^2 (m^4)	A , dy^2 (m^4)	Ix_0 (m^4)	Iy_0 (m^4)
1	0.16	0	0.9	0	0.1296	0.00053	0.00853
2	0.12	0	0.5	0	0.03	0.0036	0.0004
3	0.12	0	0.1	0	0.0012	0.0004	0.0036
Σ	0.4			0	0.1608	0.00453	0.01253

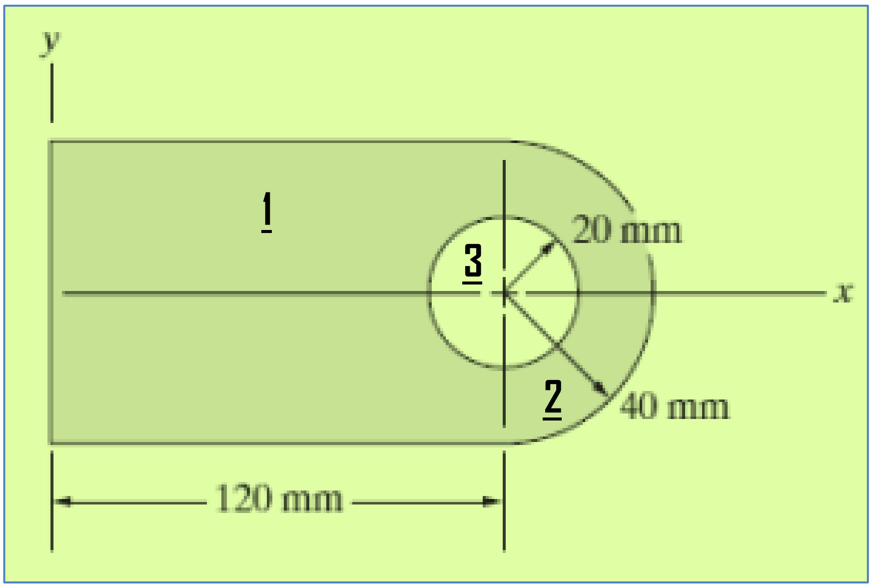
$$I_x = \Sigma I_{x0} + \Sigma A\,.\,d_y^2 = 0.00453 + 0.1608 = 0.1733\ \mathrm{m}^4 = 173.3\ (10^9)\ mm^4$$

$$I_y = \Sigma I_{y0} + \Sigma A\,.\,d_x^2 = 0.01253 + 0 = 0.01253\ m^4 = 125.3\ (10^9)\ mm^4$$

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{0.1733}{0.4}} = 0.43325\ m = 433.25\ mm$$

$$k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{0.01253}{0.4}} = 0.177\ m = 177\ mm$$

Q_2: Determine the moment of inertia of the section relative to the x-axis?



A) I_x = 6.0 (10^6) mm^4
B) I_x = 9.0 (10^6) mm^4
C) I_x = 12.0 (10^6) mm^4
D) I_x = 15.0 (10^6) mm^4

Solution:

$$A = w\,.\,L \;\;;\;\; A = \pi\,.\,r^2 \;\;;\;\; x = 0.424\,r$$

$$I_{x_o} = \frac{1}{12} bh^3 \;;\;\; I_{x_o} = I_{y_o} = \frac{\pi d^4}{64} \;\;;\;\; I_{y_o} = \frac{(9\pi^2-64)d^4}{72\pi} = 0.1098\,r^4 \;;\; I_{x_o} = \frac{\pi r^4}{8}$$

Part	A (mm^2)	dx (m)	dy (mm)	A . dx^2 (m^4)	A . dy^2 (m^4)	Ix_0 (m^4)	Iy_0 (m^4)
1	9600	60	0	34560000	0	5120000	138240000
2	2514.4	136.96	0	47165219.8	0	1005760	281088
3	-1257.2	120	0	-18103680	0	-125720	-125720
Σ	10857.2			63621539.8	0	6000040	138395368

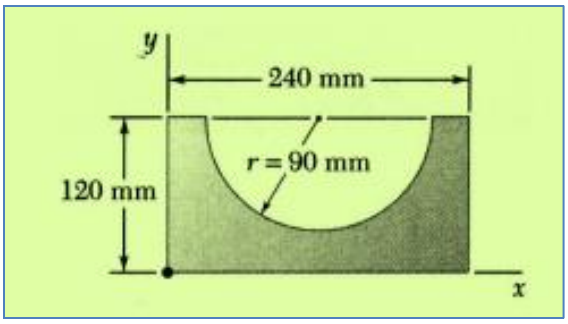
$$I_x = \Sigma I_{x0} + \Sigma A\,.\,d_y^2 = 6000040 + 0 = 6\,(10^6)\;mm^4$$

$$I_y = \Sigma I_{y0} + \Sigma A\,.\,d_x^2 = 138395368 + 63621539.8 = 202.02\,(10^6)\;mm^4$$

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{6000040}{10857.2}} = 23.51\;mm$$

$$k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{202016907}{10857.2}} = 136.41\;mm$$

Q$_3$: Determine the moment of inertia of the shaded area with respect to the x - axis?

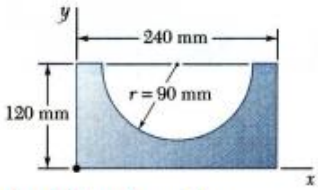


Solution:

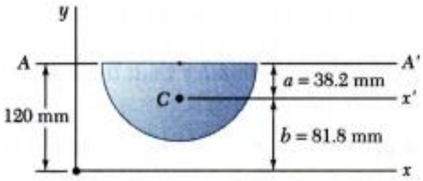
$$A = w\,.L \quad ; \quad A = \pi\,.r^2 \quad ; \quad x = 0.424\,r$$

$$I_{x_o} = \frac{1}{12}bh^3\,; \quad I_{x_o} = I_{y_o} = \frac{\pi d^4}{64} \quad ; \quad I_{x_o} = \frac{(9\pi^2 - 64)d^4}{72\pi} = 0.1098\,r^4\,; I_{y_o} = \frac{\pi r^4}{8}$$

Area Moments of Inertia



Example: Solution



$$a = \frac{4r}{3\pi} = \frac{(4)(90)}{3\pi} = 38.2\,mm$$

$$b = 120 - a = 81.8\,mm$$

$$A = \tfrac{1}{2}\pi r^2 = \tfrac{1}{2}\pi(90)^2$$

$$= 12.72\times10^3\,mm^2$$

<u>SOLUTION</u>:

- Compute the moments of inertia of the bounding rectangle and half-circle with respect to the *x* axis.

Rectangle:

$$I_x = \tfrac{1}{3}bh^3 = \tfrac{1}{3}(240)(120)^3 = 138.2\times10^6\,mm^4$$

Half-circle:

moment of inertia with respect to *AA'*,

$$I_{AA'} = \tfrac{1}{8}\pi r^4 = \tfrac{1}{8}\pi(90)^4 = 25.76\times10^6\,\text{mm}^4$$

Moment of inertia with respect to *x'*,

$$\bar{I}_{x'} = I_{AA'} - Aa^2 = (25.76\times10^6) - (12.72\times10^3)(38.2)^2$$

$$= 7.20\times10^6\,mm^4$$

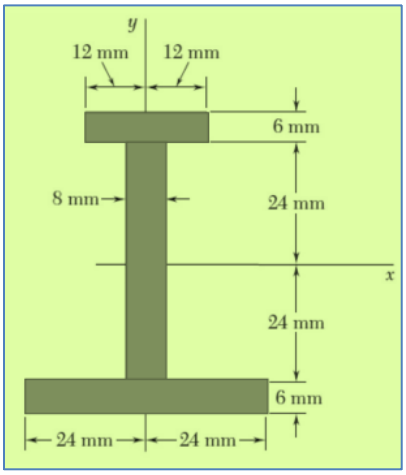
moment of inertia with respect to *x*,

$$I_x = \bar{I}_{x'} + Ab^2 = 7.20\times10^6 + (12.72\times10^3)(81.8)^2$$

$$= 92.3\times10^6\,mm^4$$

{*Answer:* $I_x = 92.3\times10^6 m^4$ }

Q$_4$: Determine the moment of inertia of the area shown in the figure?

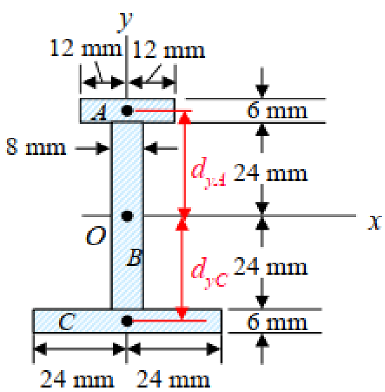


Solution:

$$\boldsymbol{A = w\,.L}$$

$$\boldsymbol{I_{x_o}\frac{1}{12}bh^3 \;;\; k_x = \sqrt{\frac{I_x}{A}} \quad ;\; k_y = \sqrt{\frac{I_y}{A}}}$$

SOLUTION



$$I_x = (\bar{I}_x + Ad_y^2)_A + (\bar{I}_x + Ad_y^2)_B + (\bar{I}_x + Ad_y^2)_C$$

(red annotation: Ad_y^2 of B → 0)

$$= [\frac{1}{12}(24)(6)^3 + (24\times 6)(27)^2]_A$$

$$+[\frac{1}{12}(8)(48)^3 + 0]_B$$

$$+[\frac{1}{12}(48)(6)^3 + (48\times 6)(27)^2]_C$$

$$I_x = 390\text{x}10^3 \text{ mm}^4$$

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{390\times 10^3}{[(24\times 6)+(8\times 48)+(48\times 6)]}} = 21.9\ mm$$

$$I_y = (\bar{I}_y + Ad_x^2)_A + (\bar{I}_y + Ad_x^2)_B + (\bar{I}_y + Ad_x^2)_C$$

(red annotations: each Ad_x^2 → 0)

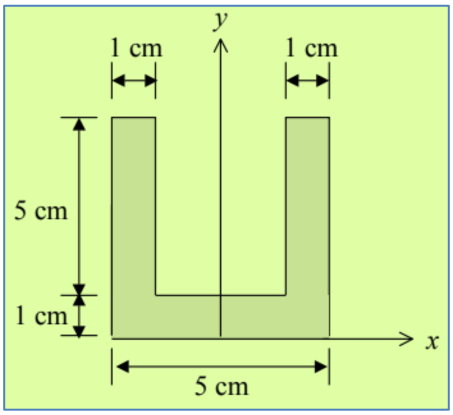
$$= [\frac{1}{12}(6)(24)^3]_A + [\frac{1}{12}(48)(8)^3]_B + [\frac{1}{12}(6)(48)^3]_C$$

$$I_y = 64.3\text{x}10^3 \text{ mm}^4$$

$$k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{64.3\times 10^3}{[(24\times 6)+(8\times 48)+(48\times 6)]}} = 8.87\ mm$$

{Answer: $I_x = 39 \times 10^4\ mm^4$ }

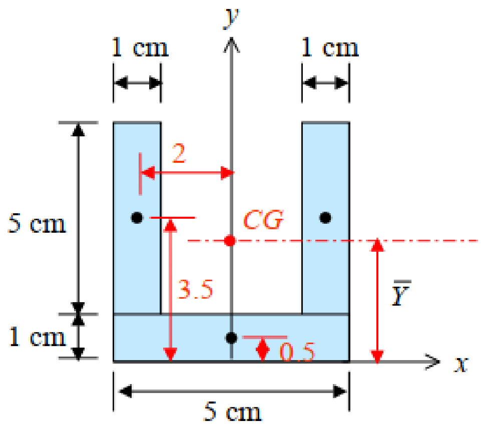
Q_5: Determine the moments of inertia and the radius of gyration of the shaded area with respect to the x and y axes and at the centroid axes?



Solution:

$$A = w\,.L$$

$$I_{x_o}\frac{1}{12}bh^3 \;;\; k_x = \sqrt{\frac{I_x}{A}} \quad ;\; k_y = \sqrt{\frac{I_y}{A}}$$



$$\bar{Y}\sum A = \sum \bar{y}A$$

$$\bar{Y} = \frac{2[(3.5)(5\times1)]+(0.5)(1\times5)}{3(5\times1)}$$

$$= 2.5\ cm$$

- **Moments of inertia about *x* axis**

$$I_x = 2[(\frac{1}{12}(1)(5)^3+(5\times1)(3.5)^2]+\frac{1}{3}(5)(1)^3$$

$$= 145\ cm^4$$

- **Moments of inertia about centroid**

$$\bar{I}_x = I_x - Ad_y{}^2$$

$$= 145-(15)(2.5)^2$$

$$= 51.25\ cm^4$$

OR

$$\bar{I}_x = 2[(\frac{1}{12}(1)(5)^3+(5\times1)(1)^2]$$

$$+[(\frac{1}{12}(5)(1)^3+(5\times1)(2)^2]$$

$$= 51.25\ cm^4$$

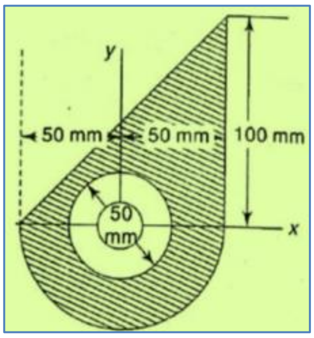
$$\bar{I}_y = I_y = 2[(\frac{1}{12}(5)(1)^3+(5\times1)(2)^2]+\frac{1}{12}(1)(5)^3$$

$$= 51.25\ cm^4$$

$$\bar{k}_x = \bar{k}_y = \sqrt{\frac{\bar{I}_x}{A}} = \sqrt{\frac{51.25}{15}} = 1.848\ cm$$

{*Answer:* $I_x = 145\ cm^4$, $\breve{I}_x = 51.25\ cm^4$, $\breve{I}_y = 51.25\ cm^4$, $\breve{K}_x = \breve{K}_y = 1.848\ cm$}

Q$_6$: : Determine the moment of inertia of the area and the radius of gyration shown in the figure?



Solution:

$$A = w\,.L \quad ; \quad A = \pi\,.r^2 \quad ; \quad x = 0.424\,r$$

$$A_T = A_1 + A_2 - A_3$$

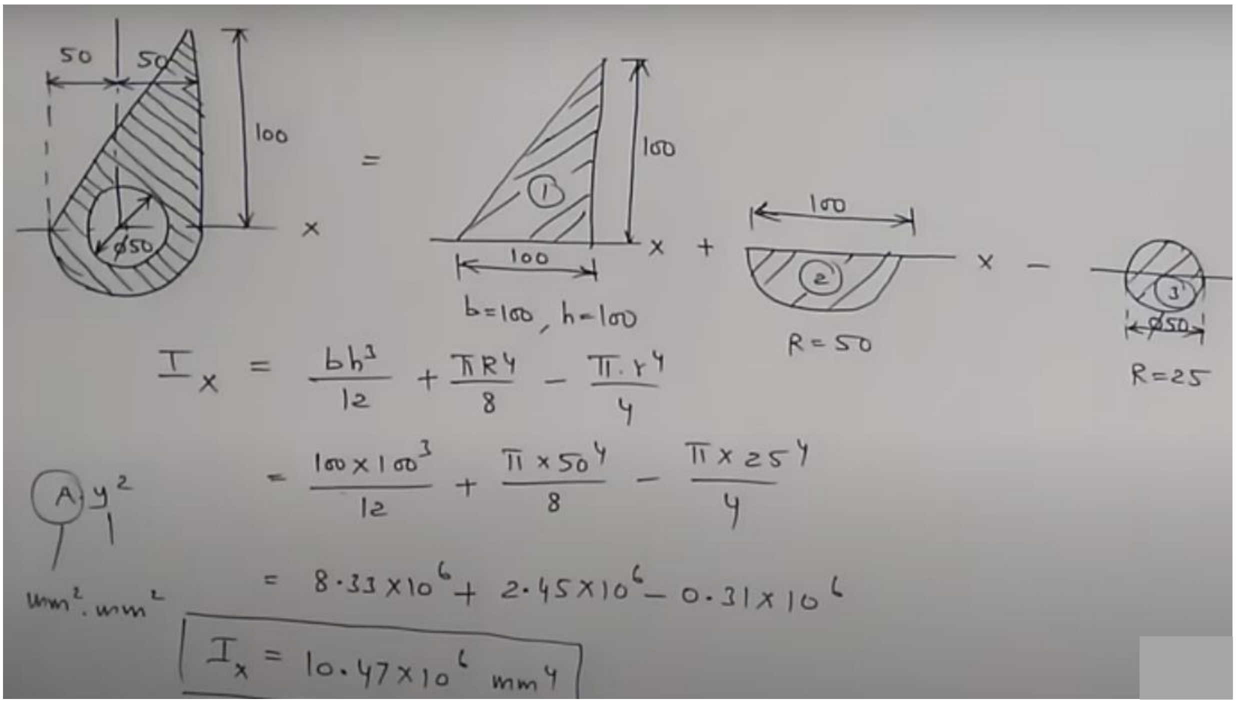
$$A_T = \frac{bh}{2} - \frac{\pi d^2}{8} - \frac{\pi d^2}{4}$$

$$A_T = \frac{100 \times 100}{2} + \frac{3.143 \times (100)^2}{8} - \frac{3.143 \times (50)^2}{4}$$

$$A_T = 5000 - 3928.75 - 1964.375 = 89\ mm^2$$

$$I_{x_o} = \frac{1}{12} bh^3 \,; \ I_{x_o} = I_{y_o} = \frac{\pi d^4}{64} \quad ; \ I_{x_o} = \frac{(9\pi^2 - 64)d^3}{72\pi} = 0.1098\,r^4 \,; I_{y_o} = \frac{\pi r^4}{8}$$

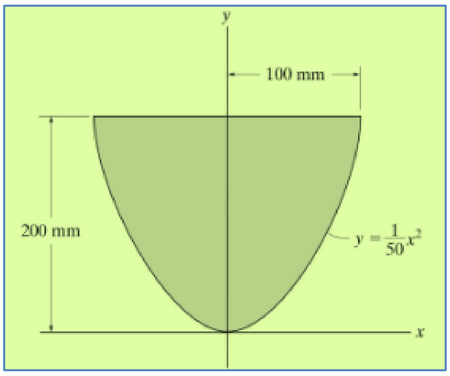
$$k_x = \sqrt{\frac{I_x}{A}} \quad ; \ k_y = \sqrt{\frac{I_y}{A}}$$

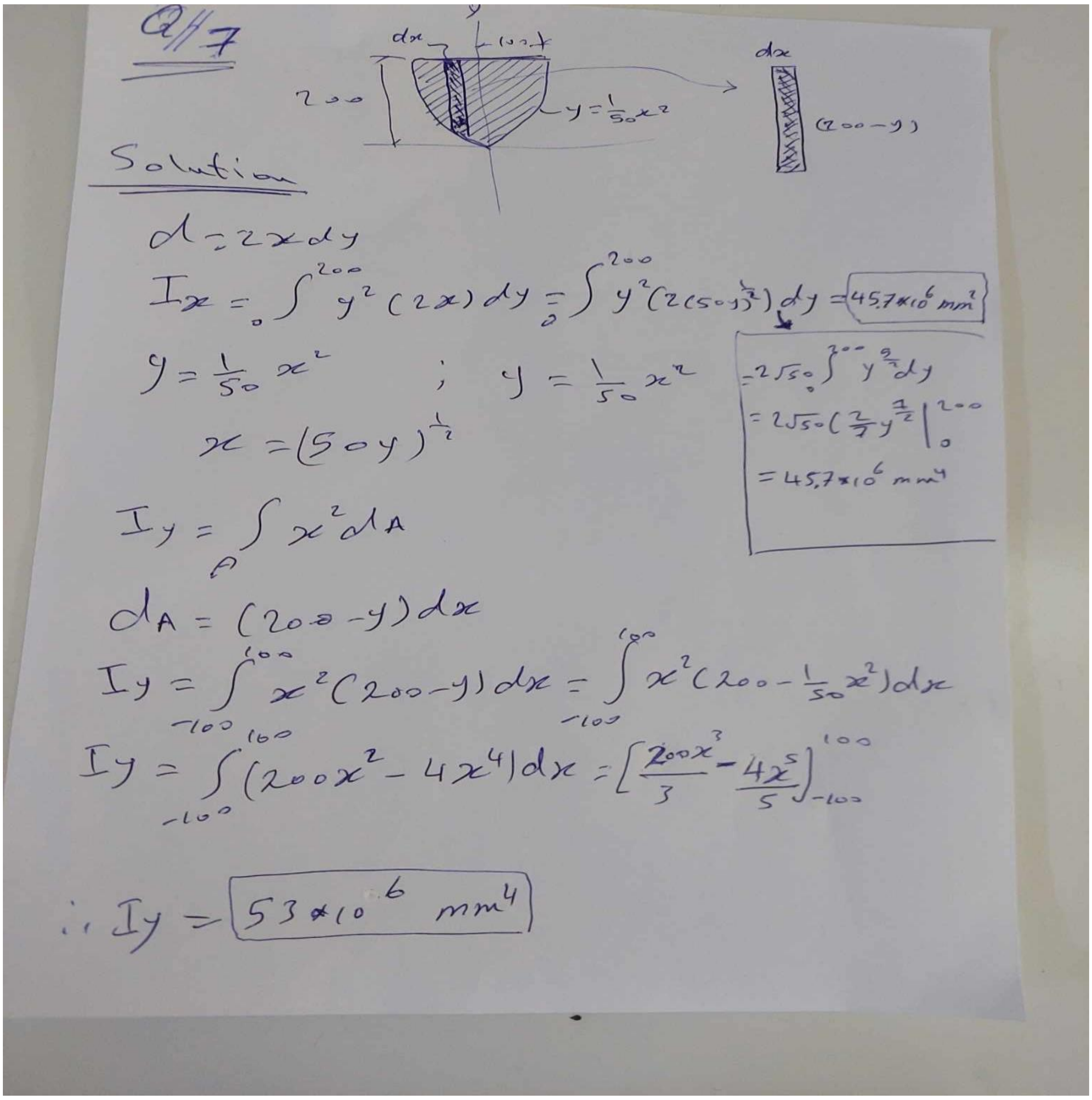


{*Answer:* $I_x = 10.47 \times 10^6\ mm^4$ }

Q_7: Determine the moment of inertia of the shaded area with respect to the x – axis and y-axis?

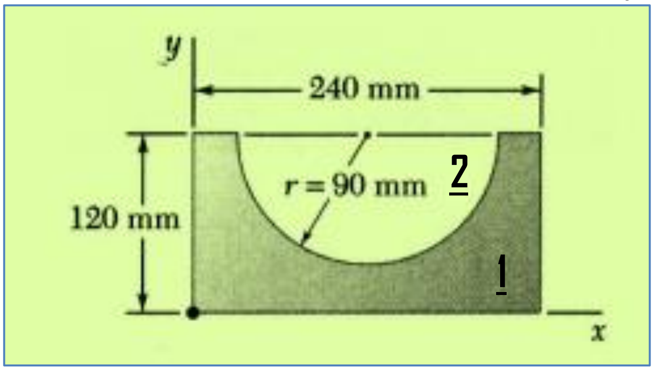
$$y = \frac{1}{50}\, x^2$$





{*Answer:* $I_x = 45.7 \times 10^6\ mm^4$, $I_y = 53 \times 10^6\ mm^4$}

Q3: Determine the moment of inertia of the shaded area with respect to the x - axis?



Solution:

$$A = w\,.\,L \quad ; \quad A = \pi\,.\,r^2 \quad ;$$

$$x = 0.424\,r = 38.16\,mm$$

$$I_{x_o} = \frac{1}{12}bh^3\,;\; I_{x_o} = I_{y_o} = \frac{\pi d^4}{64} \quad ;\; I_{y_o} = \frac{(9\pi^2-64)d^4}{72\pi} = 0.1098\,r^4\,;\, I_{x_o} = \frac{\pi r^4}{8}$$

Part	A (m^2)	dx (m)	dy (mm)	A , dx^2 (m^4)	A , dy^2 (m^4)	Ix_0 (m^4)	Iy_0 (m^4)
1	28800	120	60	864000	216000	34560000	138240000
2	-25458.3	120	81.84	1178496	548146.8	-25776528.75	
Σ	3341.7			2042496	764146.8	8783471.25	

$$I_x = \Sigma I_{x0} + \Sigma A\,.\,d_y^2 = 8783471.25 + 764146.8 = 9547618 \text{ mm}^4$$

$$I_y = \Sigma I_{y0} + \Sigma A\,.\,d_x^2 = 0.01253 + 0 = 0.01253\,m^4 = 125.3\,(10^9)\,mm^4$$

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{0.1733}{0.4}} = 0.43325\,m = 433.25\,mm$$

$$k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{0.01253}{0.4}} = 0.177\,m = 177\,mm$$

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